C. INTEGRAL CALCULUS

9. MOTIVATING EXAMPLES FOR THE CONCEPT OF AN INTEGRAL

* motion of a point particle: in the lecture

* work (physics): at home

* area under a curve: in the lecture

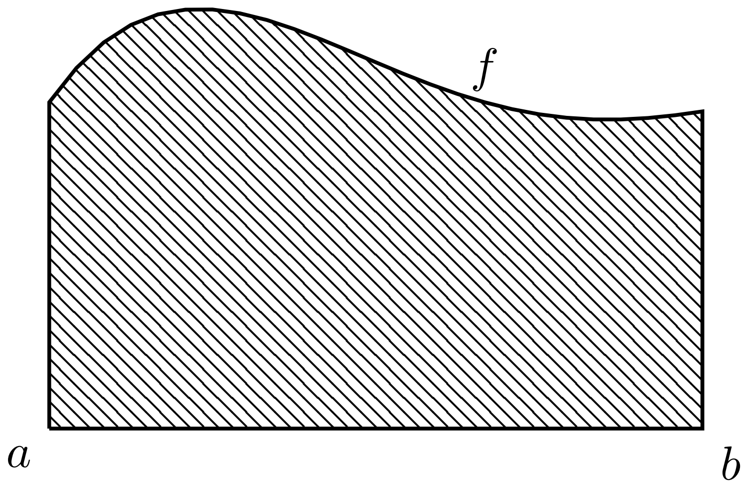
* volume of a solid of rotation: at home

(9.2) Motion of a point particle

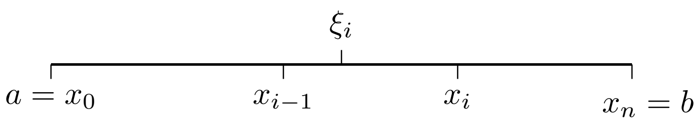
Now let's consider such a subinterval more closely:

In section (9.3) of the textbook, the concept of physical work is treated analogously; please read this section yourself at home.

(9.4) Area



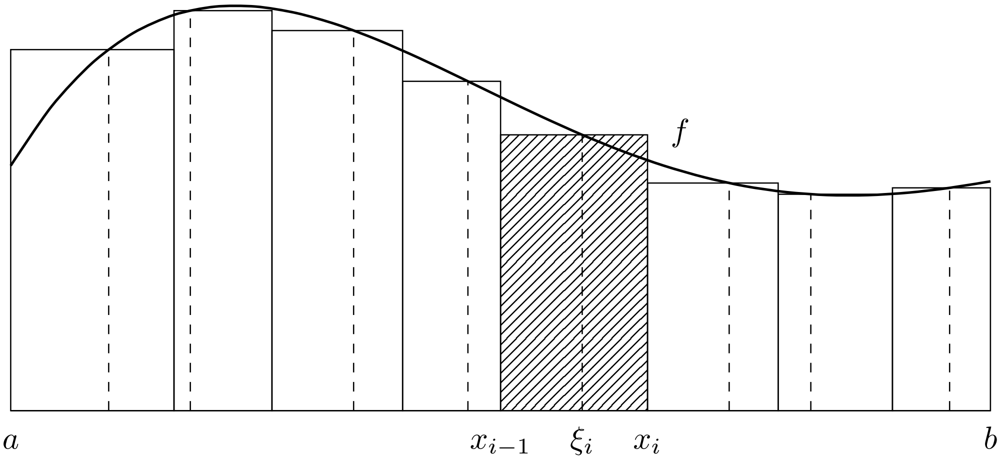
Let $f:[a,b]\to\mathbb{R}$ be a continuous function with $f(x)\geq 0$ for all $x\in[a,b]$. We would like to determine the area A of the shaded region. With this in mind, we divide $[a,b]$ into subintervals as before and in every $[x_{i-1},x_i]$ we choose an intermediate point ξ_i.



The expression $f(\xi_i)\Delta x_i$ has a straightforward meaning here: it is simply the area of the shaded rectangle in the figure below. The sum

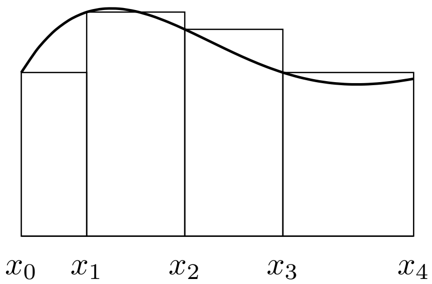
$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

is thus the area of the region composed of all the individual rectangles, which makes it an approximation of the area A:

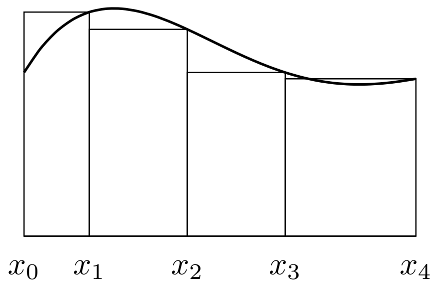


Thanks to this geometric interpretation, it's particularly clear here what happens when we choose the intermediate points differently. We sketch four possibilities:

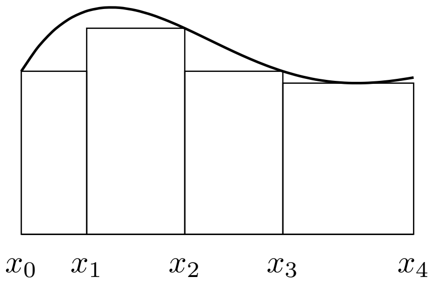
1) $\xi_i = x_{i-1}$



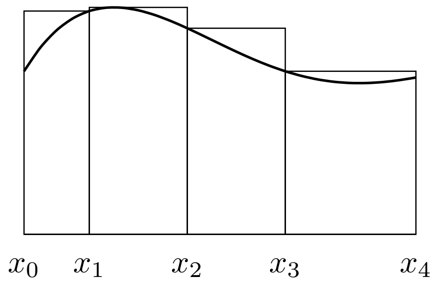
2) $\xi_i = x_i$



3) $f(\xi_i)$ the smallest value of f in $[x_{i-1}, x_i]$ ("lower sum")



4) $f(\xi_i)$ the largest value of f in $[x_{i-1}, x_i]$ ("upper sum")



In order to obtain the desired area A, we will now – as we did in the preceding examples – allow the length Δx_i of the subintervals to approach 0. We find that

$$A = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)\,dx\ .$$

In section (9.5) of the textbook, the volume of a solid of rotation is treated analogously; please read this section yourself at home.

(9.6) Summary

In the four examples discussed in this chapter we have seen that the same mathematical construction appears in very different physical settings: namely a limit which (up to the choice of notation) always takes the form

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \,.$$

We denote this limit by the abbreviation

$$\int_a^b f(x)\,dx \,.$$

These four cases share the following essential characterisation:

We are looking for a mathematical description of a quantity (whether derived from the natural sciences or from mathematics) of which the following is true:

- It can be approximated by the sum of many small sub-quantities.
- The smaller (and therefore the more numerous) these sub-quantities are, the better the approximation they yield.

Under these conditions, the desired quantity is the limit of these approximating sums. A limit of this kind is called a definite integral.

Since this chain of reasoning also recurs in many other settings, it will be useful to carry out the construction described here once and for all, independently of any special examples. This is what we will do in the next chapter. In this way we arrive at the concept of a definite integral, which is a useful tool in many situations. This definite integral will be defined generally in the next chapter; in the following chapters we will look at how to compute it effectively as well as exploring further applications.

Important: Next, read the whole chapter of the book yourself. Certain examples (physical work, solids of rotation) could not be discussed here due to time limitations, but they may appear in the exercises or on the exam.