

This script is an extract, with gaps, from the book “Introduction to the Mathematical Treatment of the Natural Sciences I - Analysis” by Christoph Luchsinger and Hans Heiner Storrer, Birkhäuser Scripts. As a student you should buy the book as well and work your way through it completely during the course MAT 182. You are allowed to save this PDF and modify it as you like, for your own use during MAT 182. For further use outside of MAT 182, please contact the lecturer, Christoph Luchsinger of the University of Zürich, in advance. The copyright remains with Birkhäuser!

8. THE DERIVATIVE OF A VECTOR FUNCTION

(8.2) Vector functions

Preliminary remarks:

At the time t , the particle is located at the point R . Its position is thus defined by the vector

$$\vec{r} = \overrightarrow{OR} .$$

Since the point R and therefore the vector $\vec{r} = \overrightarrow{OR}$ change over time, instead of \vec{r} we should really write

$$\vec{r}(t) .$$

So we are considering the vector \vec{r} as a function of one variable (in our example, of the time t). This is called a vector-valued function or a *vector function*.

In order to be able to do calculations with these vector functions, we introduce a Cartesian coordinate system as in Chapter 2. With respect to this system, then, $\vec{r}(t)$ has the coordinates

$$\vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

The coordinates $x_i(t)$ ($i = 1, 2, 3$) are dependent on the time t just as the vector $\vec{r}(t)$ is. Since $x_i(t)$ is always a real number, the *coordinate functions* (also referred to as component functions) $x_1(t), x_2(t), x_3(t)$ are just ordinary functions of one variable.

In this way, a vector function $\vec{r}(t)$ is expressed through three real-valued functions. Instead of $x_1(t), x_2(t), x_3(t)$ one can also write $x(t), y(t), z(t)$.

Examples

2. We consider the vector function

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}.$$

As the z -coordinate (3rd coordinate) here is equal to 0, the vector $\vec{r}(t)$ lies in the x - y -plane. (One could also just omit the 3rd coordinate; with a view to example 4 below, we won't do that here!) Applying the familiar formula $\sin^2 t + \cos^2 t = 1$, we see that the points with coordinates $x = \cos t, y = \sin t$ must lie on the unit circle. It follows that the vector $\vec{r}(t)$ describes a *circular motion* (counterclockwise) of a particle in the x - y -plane. At time $t = 0$ the particle is located at the point $(1,0)$, and again at time $t = 2\pi$ after it has completed a full revolution (and analogously at every point in time $t = 2n\pi, n \in \mathbb{Z}$).

4. Finally, let

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

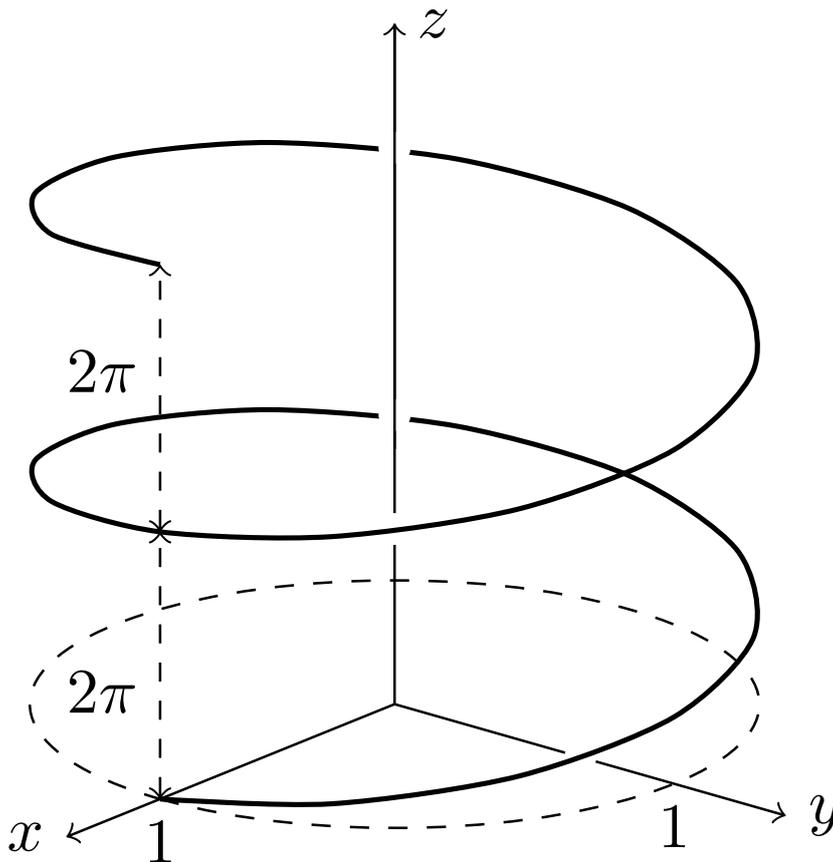
In contrast to 2. above, here the vector $\vec{r}(t)$ does not necessarily lie in the x - y -plane, but rather its endpoint at time t has a “height” (z -coordinate) of t .

The projection of $\vec{r}(t)$ onto the x - y -plane though is still the unit circle, exactly as in Example 2.

At time $t = 0$ we have $\vec{r}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

At time $t = 2\pi$ we have $\vec{r}(2\pi) = \begin{pmatrix} 1 \\ 0 \\ 2\pi \end{pmatrix}$.

From this we realise that $\vec{r}(t)$ describes a *helix* with “pitch” 2π and “period of revolution” 2π .



Short exercises:

An additional question:

(8.3) Parametric representations of curves

Sometimes we are interested less in the motion a function describes over time and more in the curve generated by its path as a geometric entity. In this case the variable t need not be interpreted as time, and is called a “parameter”.

$$\vec{u}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}, \quad t \in [0, 2\pi]$$

is according to this interpretation *one* parametric representation of a single revolution of the helix mentioned above.

In this setting (which, you’ll note, is algebraically the same as in (8.2)) we will take an interest, not in the motion itself, but rather in the curve it has traced, the “track” of the particle. We could say this is a “static” interpretation, in contrast to the “dynamic” interpretation in (8.2).

Thus in example 2 in (8.2) it matters for the motion of such a particle whether it traverses the unit circle once or twice. Its path though is the same either way: the unit circle.

What is the difference between the following 3 formulations (all parametric representations of a line segment AB)?

$$(*) \quad \vec{u}(t) = \vec{a} + t\vec{c}, \quad t \in [0, 1]$$

$$(**) \quad \vec{v}(t) = \vec{a} + 2t\vec{c}, \quad t \in [0, \frac{1}{2}]$$

$$(***) \quad \vec{w}(t) = \vec{a} + (1 - t)\vec{c}, \quad t \in [0, 1]$$

Now we return to the topic of graphically representing curves. With a bit of skill we can sketch the graph of the curve C that is generated by a given parametrisation. Naturally this can also be performed with the help of a suitable computer program.

Examples

1. Let

$$\vec{r}(t) = \begin{pmatrix} t \\ t^2 \\ 1-t \end{pmatrix}, \quad t \in [0, 1]$$

First we consider the projection onto the x - y -plane. This is $x = x_1(t) = t$, $y = x_2(t) = t^2$, and thus $y = x^2$. So the projection of C onto the x - y -plane is a parabola. Additionally, we see that since

$$x = x_1(t) = t, \quad z = x_3(t) = 1 - t,$$

the projection of C onto the x - z -plane is a line. By first drawing this parabola and this line, we are able to make a sketch of C .

(8.4) The derivative of a vector function

We use the example of a particle's motion to motivate our definition. This movement is given by the "position function" $\vec{r}(t)$. We observe the values of this function at two points in time, t_0 and $t_0 + \Delta t$:

The difference $\Delta\vec{r} = \vec{r}(t_0 + \Delta t) - \vec{r}(t_0)$ is equal to the vector \overrightarrow{PQ} . During the time span Δt the particle has moved from P to Q (in general of course not in a straight line along the vector, but rather along some curved path). If we relate the change in position to the unit of time, i.e. we divide by Δt , then we obtain

$$\frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}.$$

This quantity is the *average velocity* (on the time interval $[t_0, t_0 + \Delta t]$). It consists of a *vector*, which shows the direction of average velocity and whose magnitude is a measure of the speed of the motion.

The expression $\frac{\Delta\vec{r}}{\Delta t}$ is nothing other than a vectorial *differential quotient* (cf. the analogue in (4.3.b)).

In order to obtain the instantaneous velocity (or simply: the *velocity*) at time t_0 , we allow Δt to approach 0 as we did in (3.2), i.e. we take the limit

$$\lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}.$$

This vector is naturally referred to as the *derivative* of the vector $\vec{r}(t)$ at the point t_0 . In other words: velocity is defined as the derivative of the position function.

Letting go of this particular physical example, we make the following general definitions:

The vector function $\vec{r}(t)$ is said to be *differentiable* at the point t_0 if the limit

$$\lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}$$

exists. This limit is called the *derivative* of $\vec{r}(t)$ at the point t_0 and is denoted by

$$\frac{d\vec{r}}{dt}(t_0) \quad \text{or} \quad \vec{r}'(t_0) \quad \text{or} \quad \dot{\vec{r}}(t_0) .$$

Let it be emphasised again: the derivative of a vector is a vector.

The two interpretations of $\dot{\vec{r}}(t_0)$:

(8.5) Computing the derivative

To compute the derivative in practice, one uses the coordinate functions. Let

$$\vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Then we have

$$\dot{\vec{r}}(t) = \lim_{\Delta t \rightarrow 0} \begin{pmatrix} \frac{x_1(t + \Delta t) - x_1(t)}{\Delta t} \\ \frac{x_2(t + \Delta t) - x_2(t)}{\Delta t} \\ \frac{x_3(t + \Delta t) - x_3(t)}{\Delta t} \end{pmatrix} = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix}.$$

We have established:

The derivative of a vector function

$$\vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

is found coordinate-wise, i.e. it is

$$\dot{\vec{r}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix}.$$

Examples

1. We analyse the helical motion

$$(\#) \quad \vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}, \quad t \in \mathbb{R}$$

2. Let's vary this example a little. The same helix (as a geometrical shape) is described by

$$(\#\#) \quad \vec{s}(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \\ 2t \end{pmatrix}, \quad t \in \mathbb{R}.$$

What is different now?

(8.6) Rules for the derivative of vector functions

Analogously to (5.2), the following rules hold for the derivative:

- (1) Sum: $(\vec{u}(t) + \vec{v}(t))' = \dot{\vec{u}}(t) + \dot{\vec{v}}(t)$
- (2) Difference: $(\vec{u}(t) - \vec{v}(t))' = \dot{\vec{u}}(t) - \dot{\vec{v}}(t)$
- (3) Multiplication by a (constant) scalar: $(r\vec{u}(t))' = r\dot{\vec{u}}(t) \quad (r \in \mathbb{R})$
- (4) Multiplication by a (scalar) function: $(r(t)\vec{u}(t))' = \dot{r}(t)\vec{u}(t) + r(t)\dot{\vec{u}}(t)$
- (5) Scalar product of vectors: $(\vec{u}(t)\vec{v}(t))' = \dot{\vec{u}}(t)\vec{v}(t) + \vec{u}(t)\dot{\vec{v}}(t)$
- (6) Vector product: $(\vec{u}(t) \times \vec{v}(t))' = \dot{\vec{u}}(t) \times \vec{v}(t) + \vec{u}(t) \times \dot{\vec{v}}(t)$

Observe that (4), (5) and (6) have exactly the same form as the usual product rule for real-valued functions. The anticommutativity of the vector product means that in (6) one must pay special attention to the order of the factors.

A short exercise: A particle moves according to

$$\vec{r}(t) = \begin{pmatrix} t \\ t^2 \\ (t-1)^2 \end{pmatrix}.$$

- a) Sketch the path for $t \in [0, 1]$.
- b) What is the magnitude of the velocity at times $t = 0$ and $t = 1$?
- c) At what point in time is the speed minimal ($t \in [0, 1]$)? How big is it then?

Continued:

A trick for finding extrema of functions where the outer function is a square root, a logarithm, or the exponential function:

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.