7. LINEARISATION AND THE DIFFERENTIAL

Why? Linearisation, because humans can only think linearly: 10 hours at 100 kilometers per hour makes 1000 kilometers ($s = vt$). A bank account grows by 2 % annually; compound interest; not linear; difficult.

Idea: replace the graph of the function by its tangent!

(7.2) The equation of the tangent

(7.3) Linearisation of a function ("fit a line")

A glance at the above sketch shows that the linear function

(1) $$\boxed{p(x) = f(x_0) + f'(x_0)(x - x_0)}$$

is a good approximation of the given function $f(x)$ for values near x_0, as the functions f and p have both the same value and the same derivative at x_0:

$$f(x_0) = p(x_0) \ ,$$
$$f'(x_0) = p'(x_0) \ .$$

To put it another way: at the value x_0 the functions f and p coincide in both their 0th and 1st derivatives. (By the 0th derivative we mean the function itself, see (4.5).)

So for values near x_0, it holds that:

(2) $$\boxed{f(x) \approx p(x)}$$

(the symbols $\approx$ and $\doteq$ are both used to mean "approximately equal").

Now a linear function like p is of course easier to work with than an arbitrary function f. Sometimes we exploit this fact by replacing $f(x)$ with $p(x)$. We say that p is a "*linear replacement function*" or that we have "*linearised*" the function f. The following table shows a few examples:

$f(x)$	$f'(x)$	x_0	$f(x_0)$	$f'(x_0)$	$p(x)$
e^x	e^x	0	1	1	$1 + x$
$\ln(x)$	$\frac{1}{x}$	1	0	1	$x - 1$
$\sqrt{1+x}$	$\frac{1}{2\sqrt{1+x}}$	0	1	$\frac{1}{2}$	$1 + \frac{1}{2}x$
$\sqrt{1+x}$	$\frac{1}{2\sqrt{1+x}}$	3	2	$\frac{1}{4}$	$2 + \frac{1}{4}(x - 3)$
$\frac{1}{1+x^2}$	$\frac{-2x}{(1+x^2)^2}$	0	1	0	1

Sketches for examples 1-2:

Sketches for examples 3-5:

Notes

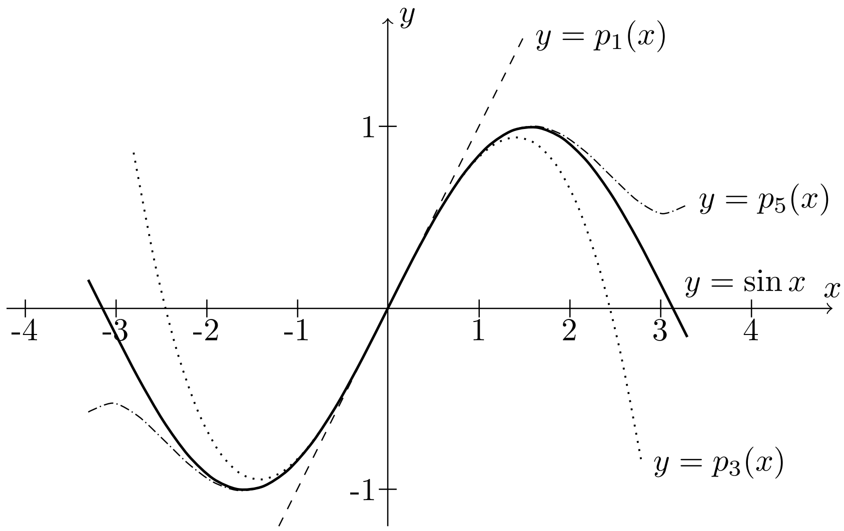
1. In the 4th example from the table, if we set $x = 3.01$ (near $x_0 = 3$), we obtain

$$f(x) = \sqrt{4.01} = 2.002498\ldots ,$$
$$p(x) = 2 + \frac{1}{4} \cdot 0.01 = 2.0025 .$$

   As we can see, the approximation is quite a good one. ⊠

2. Observe that the linear replacement function is dependent on the chosen value x_0. In the 3rd and 4th examples from the table, we linearise the same function $f(x) = \sqrt{1+x}$, but with two different values for x_0. For $x_0 = 0$ we obtain $p(x) = \frac{1}{2}x + 1$ but fo $x_0 = 3$ we get $p(x) = 2 + \frac{1}{4}(x-3) = \frac{1}{4}x + \frac{5}{4}$.
3. As the 5th example from the table shows, it can simply be the case that $p(x)$ is a constant function (this happens if and only if $f'(x_0) = 0$).

We will see in Chapter 19 that it's possible to construct replacement functions which are not merely linear functions, but also polynomials of degree n (polynomials are also relatively easy!) Here is a little illustration:



In this sketch, $y = \sin x$ and

$p_1(x) = x$,
$p_3(x) = x - \frac{x^3}{6}$,
$p_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$.

Consequences of example 2) for your bank account, and for everything else which grows exponentially (continuous time) or geometrically (discrete time):

a) of 2), p-4: $\ln x \doteq x - 1$, if $x \doteq 1$. To put it differently: if $x := 1 + h$ for a small (positive or negative) h, then it holds that:

$$\ln(1 + h) = \ln x \doteq x - 1 = 1 + h - 1 = h.$$

Thus for example we get $\ln(1.1) \doteq 0.1$ (take $h = 0.1$)
or also $\ln(0.9) \doteq -0.1$ (take $h = -0.1$), see also the sketch:

b) Bank account, interest, and compound interest:
100 CHF + 2% = 102 CHF
102 CHF + 2% $\doteq$ 104 CHF - and in fact > 104 CHF due to *compound interest.*

Analogous to this: population growth, economic growth, CO2 emissions; more in Chapter 15.

c) continuous vs discrete measurement of time:

* often both possible (annual interest vs. more frequently compounded interest)

* established notation and formulae:

discrete time:
use m, n for time
use r for the rate
formula is e.g.: $K(1 + r)^n$

continuous time:
use t, s for time
use λ for the rate
formula is e.g.: $Ke^{\lambda t}$

Doubling time and half-life:

An article of mine (in German) in the magazine Schweizer Monat:
`http://www.schweizermonat.ch/die-geheime-formel/` .

Further comments on 10 half-lives and doubling times (in a row):

(7.4) The differential

In this section we will mainly introduce some new terminology. In (7.3) we dealt with the idea of linearisation and there is not really much of substance to add. There we saw that

$$f(x) \approx p(x)$$

i.e.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \ .$$

Now we write this in a different way, namely in the form

$$f(x) - f(x_0) \approx f'(x_0)(x - x_0) \ . \tag{3}$$

If we apply our earlier abbreviations $\Delta f = f(x) - f(x_0)$ and $\Delta x = x - x_0$, this relationship takes the form

$$\Delta f \approx f'(x_0)\Delta x \ . \tag{4}$$

Finally we introduce one more notation: We let

$$\boxed{df := f'(x_0)\Delta x \ ,} \tag{5}$$

with the result that (4) becomes

$$\boxed{\Delta f \approx df \ .} \tag{6}$$

Observe that (3), (4), and (6) all convey exactly the same meaning, even though the notation is different. The following figure shows a representation of the quantities Δf and df.

Their meanings are as follows:

- Δf reflects the increase in the value of the function f as one goes from x_0 to $x = x_0 + \Delta x$.
- df represents the corresponding increase in the linear replacement function p (whose graph is the tangent line).

The quantity df is called the *differential* of f. It depends on the increase Δx as well as on the value x_0 and therefore it ought strictly speaking to be written as $df(x_0, \Delta x)$, but this is not conventionally done (compare though with example 1 below).

For formal mathematical reasons, it is also customary to write dx instead of Δx. In this way we arrive at the following formula for the differential:

$$df = f'(x_0)\, dx \; . \tag{7}$$

In ths setting dx is not an "infinitely small quantity" (whatever that might be), but rather an arbitrary number (although in practice normally a small one).

Examples

1. Let $f(x) = \sqrt{x^2+3}$. Since $f'(x) = \dfrac{x}{\sqrt{x^2+3}}$ we have $df = \dfrac{x_0}{\sqrt{x_0^2+3}}\, dx$. If we make the particular choice $x_0 = 1$, we obtain

$$df = \frac{1}{2}\, dx \; .$$

Now if we also choose a value for dx, e,g. $dx = 0.1$, it follows that

$$df = 0.05 \; ,$$

or (more precisely, although it is not normally written this way) $df(1,\, 0.1) = 0.05$. ⊠

Applications / Interpretations / An ant's-eye view of the world

(7.5) Applications to error propagation

Tedious - but important: MNF > Natural Sciences > experiments and observations > data (with errors of measurement) > calculations with data > error propagation (*the* problem: need to be able to estimate it!)

Practically: all the apparatus of Chapter 7 can be applied here; it simply acquires a new interpretation. Now x_0 becomes the true value (which does exist - we are in the natural sciences), x is the measured value (with some error); $\Delta x := x - x_0$ is the absolute error, the so-called input quantity.

Applying f to x and x_0 will lead to the propagation of error.

With Δx and f' we can decide whether a measurement error propagates badly or not (see the above sketch). As the so-called output quantity we have the true value $f(x_0)$ and the calculated value $f(x)$. $\Delta f := f(x) - f(x_0)$ is the absolute error of the function's value.

Since Δx is small, we can also replace Δf here by *df* (linear):

$$\Delta f \doteq df = f'(x_0)dx = f'(x_0)\Delta x. \qquad \text{(provisional error)}$$

In practice x_0 is unknown and so we are forced to substitute x. Thus we obtain

$$\Delta f \doteq f'(x)\Delta x. \qquad \text{(error)}$$

We do not know Δx either - but perhaps we can ascertain from the manufacturer's data or from colleagues that

$$|\Delta x| \leq a,$$

see also the following example.

Above we have introduced *absolute* error. The *relative* error of the input quantity is also defined as $\Delta x / x_0$ and the *relative* error of the output quantity as $\Delta f / f(x_0)$.

Examples

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.