6. APPLICATIONS OF THE DERIVATIVE

(6.2) Terminology

a) Intervals

Intervals are special subsets of the set of real numbers $\mathbb{R}$. If a and b $(a < b)$ are real numbers, we use the following notation:

$$
\begin{aligned}
(a, b) &= \{x \mid a < x < b\} && : \textit{open} \text{ interval,} \\
[a, b] &= \{x \mid a \leq x \leq b\} && : \textit{closed} \text{ interval,} \\
\left.\begin{aligned} (a, b] &= \{x \mid a < x \leq b\} \\ [a, b) &= \{x \mid a \leq x < b\} \end{aligned}\right\} && : \textit{half-open} \text{ intervals.}
\end{aligned}
$$

Similarly, we write for example

$$
\begin{aligned}
(a, \infty) &= \{x \mid x > a\}, \\
(-\infty, b] &= \{x \mid x \leq b\} \\
(-\infty, \infty) &= \mathbb{R} \qquad \text{etc.}
\end{aligned}
$$

In some books the notation $]a, b[$ is also used for an open interval.

b) ε–neighbourhoods

ε–neighbourhoods are special open intervals, where ε is a positive number (i.e. greater than 0), in practice usually a small positive number. We have already encountered such ε before (3.6.b), (4.6.d). Let $x_0 \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Then we define

$$U_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon) = \{x \mid x_0 - \varepsilon < x < x_0 + \varepsilon\} \, .$$

$x_0 - \varepsilon \qquad x_0 \qquad x_0 + \varepsilon$

This interval is called the *ε–neighbourhood* of x_0. $U_\varepsilon(x_0)$ consists in other words of all the points whose distance to x_0 is less than ε.

c) Boundary points and interior points

A half-open or closed interval has *boundary points*:

$$\begin{aligned} [a,b): &\quad \text{boundary point } a,\\ (a,b]: &\quad \text{boundary point } b,\\ [a,b]: &\quad \text{boundary points } a,b\ . \end{aligned}$$

The points in an interval I which are not boundary points are called *interior points* of I. It is clear that x_0 is an inner point of I if and only if there exists an ε–neighborhood $UU_\varepsilon(x_0)$ (possibly a small one) which is entirely contained in I.

d) Increasing and decreasing functions

Let f be a function defined on the domain $D(f)$. This function f is said to be *increasing* if the following is true: $f(x_1) < f(x_2)$ for all $x_1, x_2 \in D(f)$ with $x_1 < x_2$. Correspondingly, f is said to be *decreasing* if the following holds: $f(x_1) > f(x_2)$ for all $x_1, x_2 \in D(f)$ with $x_1 < x_2$.

Note

In the above cases one can also say that f is strictly monotonically increasing (resp. decreasing). Monotonically increasing (without the "strictly") is defined as "$f(x_1) \leq f(x_2)$ for $x_1 < x_2$"; monotonically decreasing analogously. Finally, f can also be said to be monotonic, if f is either monotonically increasing or monotonically decreasing. In this book we will stick to the simpler terminology given above.

And what happens to $f(x) = x^3$ at the point 0?

(6.3) Growth of a function

b) <u>The derivative and growth</u>

Let the function f be defined (and differentiable) on the interval I. The interval I can be open, closed, etc.; even the case $I = \mathbb{R}$ is also possible. The interval I does not necessarily have to be the "natural domain" of f though (i.e. the largest possible subset of $\mathbb{R}$ on which f could possibly be defined.) In connection with the questions we discuss here, we might wish to restrict ourselves for example to an interval on which the derivative is always positive.

If the derivative $f'(x) > 0$ for all x in I, then in graphical terms the tangent to every point of the graph has a positive slope, which means that the function is increasing on I: $x_1 < x_2$ implies $f(x_1) < f(x_2)$. A corresponding result holds in the case $f'(x) < 0$, where the function is decreasing.

We summarise as follows:

(1) $f'(x) > 0$ for all $x \in I \Longrightarrow f$ is increasing on I.

(2) $f'(x) < 0$ for all $x \in I \Longrightarrow f$ is decreasing on I.

As seen here, the "curvature of the graph" can vary; more details about that in (6.4).

Warning:

Example

Let $f(x) = e^{cx}$ and therefore $f'(x) = c \cdot e^{cx}$. Since e^{cx} is positive everywhere, it follows that

(i) For $c > 0$ we have $f'(x) > 0$: f is increasing on $\mathbb{R}$. Later we will see that we can use this to model processes which show growth over time. In such a setting, x is often replaced by t (for "**t**ime"). Examples include a growing economy or population, the early-stage spread of an epidemic, or the number of cells during cell division.

(ii) For $c < 0$ we have $f'(x) < 0$: f is decreasing on $\mathbb{R}$. With this function we will later model the decay of homogeneous radioactive material.

c) The derivative and constant functions

It is also important to consider the case where $f'(x) = 0$ for all x in the interval I. Geometrically interpreted, it becomes clear that f must then be a constant function, as its tangent is horizontal everywhere. In algebraic terms:

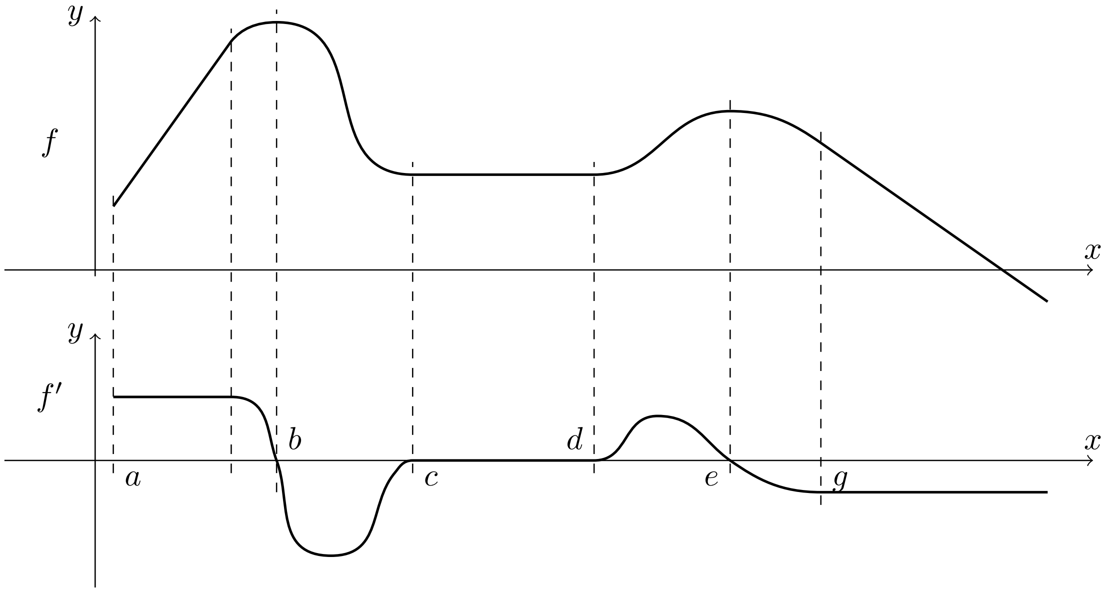
$$f'(x) = 0 \Longrightarrow f \quad \text{is constant.}$$

In contrast to b) (cf. the hint there with x^3), here the converse is also true: One of the simplest rules of derivation (5.3) states that the derivative of a constant function is always 0. Thus we have the rule

(3) $f'(x) = 0$ for all $x \in I \Longleftrightarrow f$ is constant.

d) A qualitative example

All this may be illustrated by means of a qualitative example. No claim is made to numerical accuracy!



From a to b the function increases. At first it increases steadily and so the derivative is constant, but then at the point b it becomes 0 (corresponds to a horizontal tangent!). After this the function decreases until c and is then horizontal. Between b and c, then, the derivative must be negative, but between c and d equal to 0. Next we see an increase from d to e (derivative is positive), a "stationary point" at e (derivative $= 0$) and finally a decrease (derivative is negative), during which the derivative remains constant after g (the graph is a falling line!).

(6.4) Meaning of the second derivative

In this section we will require, not merely that $f : I \to \mathbb{R}$ is differentiable, but additionally that the second derivative f'' exists on I. Since f'' is also the first derivative of f', we can apply all the observations in (6.3) to f' and f'' (instead of to f and f' as we did before). If the second derivative is negative, $f''(x) < 0$, on the whole interval I, then $f'(x)$ is decreasing on all of I, i.e. the slope of $f(x)$ decreases (as x increases) — and in the example below even becomes negative:

In visual terms, this means that the graph of f *curves to the right* (German: Rechtskurve). (Sometimes f is also said to be concave downward or concave upward, but this terminology is a little confusing.) The case $f''(x) > 0$ is completely analogous, and so we obtain:

(4)	$f''(x) < 0$ on $I \Longrightarrow$	The graph of f curves to the right on I.
(5)	$f''(x) > 0$ on $I \Longrightarrow$	The graph of f curves to the left on I.

The following sketches summarise the relationships of (4) and (5) with (1) and (2). They demonstrate the various possible combinations of growth and curvature.

Finally, one more important concept: if the graph of f changes at the point x_0 from left- curving to right-curving (or vice versa) then we say that x_0 is an *inflection point* (German: Wendepunkt). An inflection point is characterised by the conditions that $f''(x_0) = 0$ and that f'' has a sign change at x_0.

The condition $f''(x_0) = 0$ alone is not sufficient for an inflection point. Taking for example the function $f(x) = x^4$, we see that $f''(0) = 0$, but 0 is not an inflection point because the sign of f'' does not change at 0. The graph curves to the left throughout its whole length.

An inflection with a horizontal tangent (i.e. one which also satisfies $f'(x_0) = 0$) is called a *saddle point*. For example, the function $f(x) = x^3$ has a saddle point at 0.

We repeat and summarise the above definitions.

(6)	f has an inflection point at $x_0 \iff$ $f''(x_0) = 0$ and f'' has a sign change at x_0.
(7)	f has a saddle point at $x_0 \iff$ $f'(x_0) = f''(x_0) = 0$ and f'' has a sign change at x_0.

x^2 vs x^4:

Example

Analyse the function $f(x) = \frac{1}{3}x^3 - x$ with regard to the results in (6.3) and (6.4):

(6.5) Extrema

a) Introduction

One very well-known application of differential calculus is in the solution of extremum problems (the finding of a function's maxima and minima). The procedure is one which will be familiar to you from secondary school: set the first derivative equal to zero and use the second derivative as needed to check whether each solution is a maximum or a minimum. In this chapter we will examine this simple process rather more formally. We begin by more precisely stating the definition of an extremum.

b) Absolute and relative extrema (or "global" and "local" extrema)

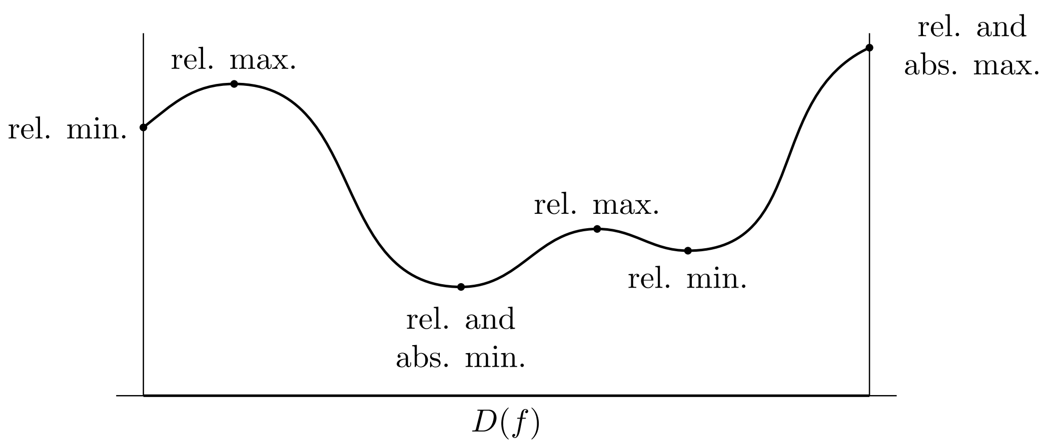
Let f be a given function that is defined on a certain domain $D(f)$. By the *absolute maximum* of f (on $D(f)$), we mean the largest value which the function takes on anywhere in the given domain. The *absolute minimum* is defined correspondingly. The maxima and minima together are called the *extrema* of f. Somewhat more formally:

Let $f : D(f) \to \mathbb{R}$ be a function. We say that f has an *absolute maximum* at x_0, or that $f(x_0)$ is an absolute maximum of f, if

$$f(x_0) \geq f(x) \quad \text{for all} \quad x \in D(f) \ .$$

The *absolute minimum* is defined in the same way (replacing $\geq$ with $\leq$).

Besides the absolute extrema, we will also find it useful to consider *relative extrema*. The following sketch illustrates these concepts.



In order to give an exact definition for this too, we need to replace the intuitive concept of "near x_0" with the more formal language of an ε–neighbourhood $U_\varepsilon(x_0)$ of the point x_0 (cf. (6.2.b)). Thus a precise formulation runs as follows:

The function $f : D(f) \to \mathbb{R}$ has a *relative maximum* at the point $x_0 \in D(f)$, if there exists an ε–neighbourhood $U_\varepsilon(x_0)$ such that

$$f(x_0) \geq f(x) \quad \text{for all} \quad x \in D(f) \cap U_\varepsilon(x_0) \ .$$

A *relative minimum* is defined correspondingly.

c) <u>On the existence of extrema</u>

Often one would like to locate the extrema (and above all the absolute extrema) of a given function $f : D(f) \to \mathbb{R}$. It is entirely possible, though, that there are none to be found. A function does not necessarily have to have any extrema, as we see by analysing the following examples:

1) $f : (0, 1] \to \mathbb{R},\ f(x) = \dfrac{1}{x}\,.$

2) $f : (0, 1) \to \mathbb{R},\ f(x) = x\,.$

d) How to find extrema

Please read this section in the textbook carefully! In the lecture, we will restrict ourselves to a short discussion of the relevant theory and follow it up immediately with some examples.

- An absolute maximum is always also a relative maximum, so we can simply search for all relative maxima - the same goes for minima.
- From $f'(x_0) = 0$ we cannot simply conclude that f has a relative extremum at x_0.
- It is also possible for extrema to exist where the derivative is not zero.

- *possible* candidates for the locations of relative extrema:

(i) Boundary points - **troublesome, frequently forgotten!**

If the domain $D(f)$ is a half-open or closed interval, then it has boundary points. The function may have an absolute or relative extremum at these points:

These boundary points are not in general captured by the methods of differential calculus (the derivative of the function f is in general not 0 at a boundary point), so if they exist, they always need to be considered separately.

(ii) Interior points at which f is differentiable - **the nice case**

Now that we have dealt with the boundary points, we consider an interior point x_0 of the domain $D(f)$ and also stipulate that f is differentiable at x_0. Then:

Let x_0 be an interior point of $D(f)$ and let f be differentiable at x_0. If f has a relative extremum at the point x_0, then $f'(x_0) = 0$.

This result is graphically obvious: if x_0 is for example a relative maximum, then the tangent line is increasing for points to the left of x_0 and decreasing for points to the right of x_0. The tangent at x_0 (which exists, according to our differentiability assumption) therefore must be horizontal: it follows that $f'(x_0) = 0$.

We will revisit it in f), but let it be emphasised right here that the converse of the above statement is not true: from $f'(x_0) = 0$ it does not necessarily follow that f has a relative extrema at x_0.

(iii) Interior points at which f is not differentiable - **you'll know if you have these!**

If f is not differentiable at x_0, the condition $f'(x_0) = 0$ is meaningless. It can happen though that an extremum occurs just where f is not differentiable. Thus the function $f(x) = |x|$ has its relative (and absolute) minimum at the point $x = 0$, where it is not differentiable! Without making any claim to completeness, care should be taken with functions which contain absolute value notation, as well as when a function is defined differently on different intervals (curly brace notation).

e) Summary

The above considerations can be summarised (with some reordering) as follows:

A *relative extremum* (if one exists at all) must appear at one of the following locations:

1. Interior points x_0 of the domain such that $f'(x_0) = 0$,
2. Boundary points of the domain (if any),
3. Points where f is not differentiable (if any).

The absolute extrema (if these exist) can be found among the relative extrema. The largest relative maximum is the absolute maximum, and the smallest relative minimum is the absolute minimum.

f) Characterisation of extrema

If the function f is twice differentiable, then the following is a helpful criterium:

> Let x_0 be a point in the domain $D(f)$ such that $f'(x_0) = 0$.
> - If $f''(x_0) < 0$, then f has a relative maximum at x_0.
> - If $f''(x_0) > 0$, then f has a relative minimum at x_0.

The rules above are seen to be plausible when we think back on the results mentioned in (6.4): a negative value for the second derivative indicates a curve which bends to the right, i.e. is open downwards, and so this corresponds to a maximum. In the same way, a positive second derivative corresponds to a minimum.

However, if $f''(x_0) = 0$, this test is not informative, and we must find out in some other way (textbook, p. 93), as shown in the following examples. In each of the following cases we have $f'(0) = 0, f''(0) = 0$.

For the record: there are really only 2 functions you need to keep in mind, to understand the contrasts at work here: x^3 ($f'(0) = 0$ but the function is increasing (even strictly monotonically increasing) for all of $\mathbb{R}$ and has no extremum, just a saddle point) and x^4 ($f'(0) = 0$ and $f''(0) = 0$, not an inflection point but a minimum).

(6.6) Examples of extremum problems

Example 1:

Example 2:

Another example:

Exercise 6-5 a):

(6.7) Graphical representation of functions

As a follow-on to this lecture, please read (6.7) in the textbook; this contains a compact summary of all the aspects of a function's behaviour which we might want to investigate. It also shows graphs of the most essential functions - here's a little memory aid `http://www.luchsinger-mathematics.ch/BeautifulDanceMoves.jpg` . A minor observation from practical experience:

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.