5. TECHNIQUES OF DIFFERENTIATION

(5.2) Rules for the derivative

Function	Derivative	Name
$f(x)+g(x)$	$f'(x)+g'(x)$	sum rule
$f(x)-g(x)$	$f'(x)-g'(x)$	difference rule
$cf(x)$	$cf'(x)$	constant-factor rule ($c \in \mathbb{R}$)
$f(x)g(x)$	$f'(x)g(x)+f(x)g'(x)$	product rule ("symmetry")
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$	quotient rule ("low (d high) - high (d low)")
$f(g(x))$	$f'(g(x)) \cdot g'(x)$	chain rule ("outer times inner")

(5.3) Derivatives of the most important functions

Function $y = f(x)$	Derivative $y' = f'(x)$	Notes
$c = \text{const}$	0	
x^n	nx^{n-1}	Valid for all $n \in \mathbb{R}$, if $x > 0$. Valid for all $n \in \mathbb{Z}$ and arbitrary x (if $n < 0$ however only for $x \neq 0$).
x	1	
$\frac{1}{x}$	$-\frac{1}{x^2}$	$x \neq 0$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$x > 0$
e^x	e^x	
a^x	$a^x \cdot \ln a$	$a > 0$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\log_a x$	$\frac{1}{x \cdot \ln a}$	$x > 0,\ a > 0$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$	$x \neq \frac{\pi}{2} + k\pi\ (k \in \mathbb{Z})$
$\cot x$	$-(1 + \cot^2 x) = -\frac{1}{\sin^2 x}$	$x \neq k\pi\ (k \in \mathbb{Z})$
$\arcsin x$	$\frac{1}{\sqrt{1 - x^2}}$	$\lvert x \rvert < 1$
$\arctan x$	$\frac{1}{1 + x^2}$	
$\arccos x$	$-\frac{1}{\sqrt{1 - x^2}}$	$\lvert x \rvert < 1$
$\operatorname{arccot} x$	$-\frac{1}{1 + x^2}$	

Computations and diagrams:

Computations and diagrams - continued:

Function $y = f(x)$	Derivative $y' = f'(x)$	Notes
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\tanh x$	$1 - \tanh^2 x = \dfrac{1}{\cosh^2 x}$	
$\coth x$	$1 - \coth^2 x = -\dfrac{1}{\sinh^2 x}$	$x \neq 0$
$\operatorname{arsinh} x$	$\dfrac{1}{\sqrt{1+x^2}}$	
$\operatorname{arcosh} x$	$\dfrac{1}{\sqrt{x^2-1}}$	$x > 1$
$\operatorname{artanh} x$	$\dfrac{1}{1-x^2}$	$\lvert x\rvert < 1$
$\operatorname{arcoth} x$	$\dfrac{1}{1-x^2}$	$\lvert x\rvert > 1$

(5.4) Examples

1. $f(x) = x^4 + 6x^3 + 2.$

2. $G(t) = \dfrac{2}{t^2} + \sqrt[3]{t} - \dfrac{3}{\sqrt[4]{t^3}}, \ (t > 0).$

3. $\Phi(z) = z \ln z, \ (z > 0).$

4.1 Preparation for 4.2: take the derivative of $\frac{1}{x}$ using the quotient rule...

4.2 $A(\alpha) = \dfrac{\sin\alpha}{1 + \cos\alpha}\,.$

(5.5) Details of the chain rule (in German: Kettenregel)

a) <u>Compositions of functions (German: zusammengesetzte Funktionen)</u>

Now we can define generally what we mean by the composition of two functions:

Let $f : D(f) \to \mathbb{R}$ and $g : D(g) \to \mathbb{R}$ be two functions, and let $D(g)$ be chosen so that $y = g(x) \in D(f)$ for all $x \in D(g)$.
Then for every $x \in D(g)$ we can form the number

$$f(g(x)) \quad (\text{or} \quad f(y) \quad \text{with} \quad y = g(x)) \; .$$

The function thus obtained,

$$h : D(g) \to \mathbb{R},\; h(x) = f(g(x)) \; ,$$

is called the *composition* (German: Zusammensetzung, Komposition, Verkettung) of g and f. In symbols:

$$h = f \circ g \; .$$

We call g the *inner function* and f the *outer function*.

<u>Example 3</u>

The function

$$h(x) = e^{x^2}$$

is the composition of $y = g(x) = x^2$ and $f(y) = e^y$:

$$h(x) = f(g(x)) = e^{x^2} \ .$$

It is defined for all $x \in \mathbb{R}$. ☒

<u>Example 4</u>

The function

$$k(x) = \sqrt{x^2 - 1}$$

is the composition of $y = g(x) = x^2 - 1$ and $f(y) = \sqrt{y}$:

$$k(x) = f(g(x)) = f(y) = \sqrt{x^2 - 1} \ .$$

It is defined for all x for which $x^2 - 1 \geq 0$, i.e. for all x with $|x| \geq 1$. ☒

<u>Example 5</u>

Naturally it is also possible to compose more than two functions. So for instance

$$p(x) = \sin(\ln(x^2 + 1))$$

is of the form $p(x) = f(g(h(x)))$ with $f(z) = \sin z$, $g(y) = \ln y$ and $h(x) = x^2 + 1$. ☒

b) <u>Application of the chain rule</u>

This rule tells how to form the derivative of the composite function $f \circ g$, i.e. the function $x \mapsto f(g(x))$. The formula states (cf. (5.2)):

$$\boxed{f(g(x))' = f'(g(x)) \cdot g'(x)}$$

or in words:

> Take the derivative of the function f at the point $y = g(x)$ and multiply this with the derivative of g at the point x.

$f'(y) = f'(g(x))$ is referred to as the *outer*, $g'(x)$ as the *inner derivative*. In short, then, the rule can be stated as:

outer derivative times inner derivative.

(This presupposes naturally that both $f'(g(x))$ and $g'(x)$ exist in the first place. For the precise formulation, see the proof of this rule in (27.3).)

Examples 6-10

Examples 6-10, continued

Before Examples 11-12, **Rules of the logarithm and exponential functions**:

Before Examples 11-12, **Rules of the logarithm and exponential functions**: (continued)

Before Examples 11-12, **Rules of the logarithm and exponential functions**:
(continued again)

Example 11

Example 12

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.