23. DIFFERENTIAL CALCULUS OF MULTIVARIATE FUNCTIONS

(23.2) Partial derivatives

a) Initial examples

If we have a function of multiple variables, we can take the derivative with respect to each of these variables individually. In this way we obtain the so-called partial derivatives of this function. We illustrate this at first by considering some functions of two variables. As in (22.6) we can hold one variable or the other constant and thus form two partial functions:

$$\varphi(x) = f(x, y_0) \ , \qquad y_0 \quad \text{constant} \ ,$$
$$\psi(y) = f(x_0, y) \ , \qquad x_0 \quad \text{constant} \ .$$

As φ and ψ are now functions of a single variable, we can take their derivatives in the usual way.

Examples

b) General definition and notation

An informal definition can be given in words as follows (for a precise formulation see d)):

> The partial derivative of f with respect to x is the usual derivative of the partial function of f in the x-direction.

An analogous definition can of course be made for y and — if there are more than two variables — also for all the remaining variables.

Different notations for the partial derivative are in use. For the partial derivatives at the point $x = x_0,\ y = y_0$ one writes

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad f_x(x_0, y_0)\ ,$$
$$\frac{\partial f}{\partial y}(x_0, y_0) \quad \text{or} \quad f_y(x_0, y_0)\ .$$

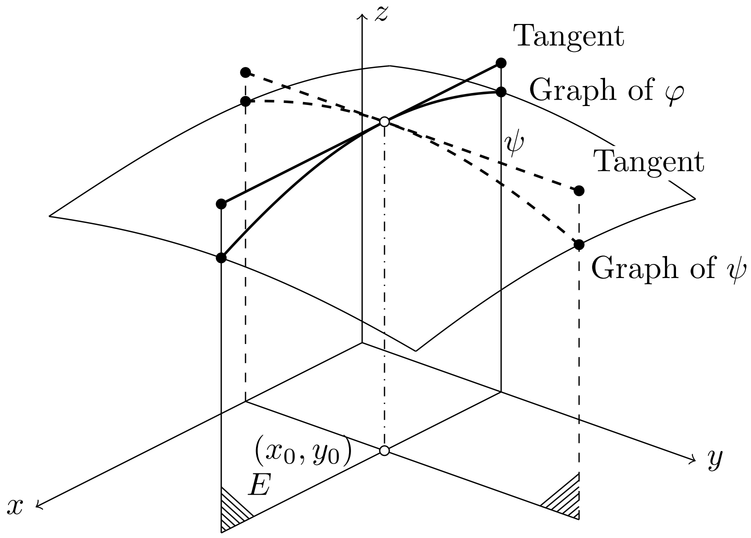
The rounded ∂ are meant to signify that these are partial derivatives and not the usual derivative.

c) Practical computation: examples

We recall that the definition of a partial function consisted of holding all variables but one constant. From that, the following practical rule results:

> To take the partial derivative of f with respect to some variable, e.g. x, we can mentally regard all variables except x as constants and then apply the usual rules for the derivative with regard to the only remaining variable x.

Geometric interpretation of partial derivatives



Examples

(23.4) Higher partial derivatives

Let f be a function which is partially differentiable on D. Then as mentioned previously we can consider the partial derivatives as functions:

$$f_x : D \to \mathbb{R} \ , \quad f_y : D \to \mathbb{R} \ .$$

These functions may themselves be partially differentiable, i.e. we may be able to take their partial derivatives with respect to x resp. y. The following are possible:

- derivative of f_x with respect to x: result is $f_{xx}(x,y)$ or $\dfrac{\partial^2 f}{\partial x^2}(x,y)$,
- derivative of f_x with respect to y: result is $f_{xy}(x,y)$ or $\dfrac{\partial^2 f}{\partial y \partial x}(x,y)$.
- derivative of f_y with respect to x: result is $f_{yx}(x,y)$ or $\dfrac{\partial^2 f}{\partial x \partial y}(x,y)$,
- derivative of f_y with respect to y: result is $f_{yy}(x,y)$ or $\dfrac{\partial^2 f}{\partial y^2}(x,y)$.

The resulting functions are naturally called the *second partial derivatives* of f. The procedure for functions of more than two variables is completely analogous. We repeat the notations just introduced:

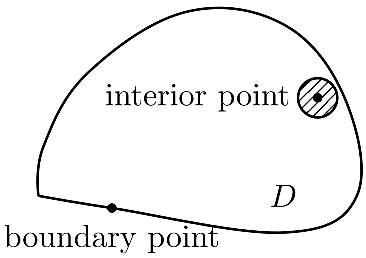
- partial derivative of f with respect to x twice: $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$.
- partial derivative of f first with respect to x, then y: $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$.

In the second case the order of x and y differs according to the notation used. As we will see below, however, this is irrelevant in most cases.

Examples

(23.5) Extrema of bivariate functions

a) Preliminaries At this point in the textbook such terms as ϵ-neighbourhoods in $\mathbb{R}^2$, interior points of a set,



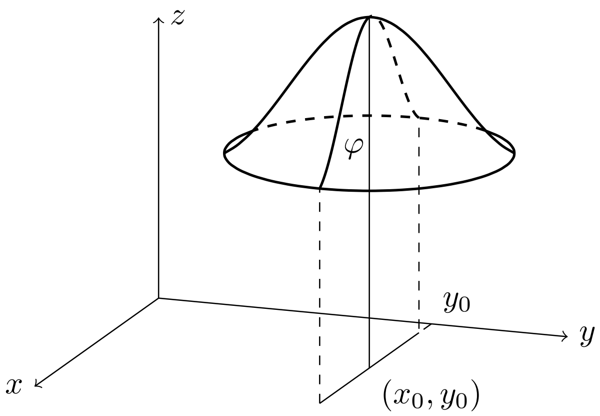
absolute and relative extrema are given precise definition. As these concepts have already been discussed in one dimension and are intuitively quite clear, in the interest of time we will refer to the book and will discuss these concepts here purely through some illustrative examples.

b) How do we compute extrema?

In analogy to the univariate case (6.5) the following is a necessary condition:

> Let (x_0, y_0) be an interior point of $D(f)$ and f a function which is partially differentiable here. If f has a relative extremum at (x_0, y_0), then it must hold that
>
> $$f_x(x_0, y_0) = 0 \;, \qquad f_y(x_0, y_0) = 0 \;.$$

We give an informal justification for the case of a relative maximum:



If f has a relative maximum at (x_0, y_0), then the partial function

$$\varphi(x) = f(x_0, y_0)$$

also has a relative maximum at x_0. It follows from the necessary condition in (6.5.d)) for univariate functions that then $\varphi'(x_0) = 0$. Here though $\varphi'(x_0) = f_x(x_0, y_0)$ (vgl.

(23.2.d)), from which the condition

$$f_x(x_0, y_0) = 0$$

follows at once. The second condition

$$f_y(x_0, y_0) = 0$$

is proven completely analogously. We can also give an analogue to the summary in (6.5.d)):

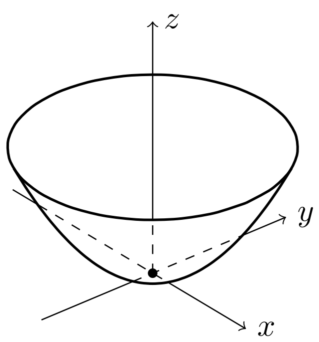
A relative extremum (if one exists at all) occurs at one of the following locations:
1. Interior points (x_0, y_0) with $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$,
2. Boundary points of $D(f)$,
3. Points at which f is not partially differentiable.

Among these possible candidates for extrema one obtains through comparison the absolute maxima and minima, provided these exist. Please read and take note of the warnings in the textbook about multivariate functions on pages 358-359!

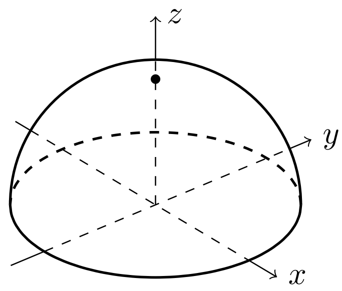
c) How can we determine what sort of extremum we have?

Even in the univariate case, the condition that the derivative is equal to zero is not a sufficient condition for an extremum. We can expect, therefore, that in the case of two variables this will be no different. We investigate a few examples (cf. (22.4.b)):

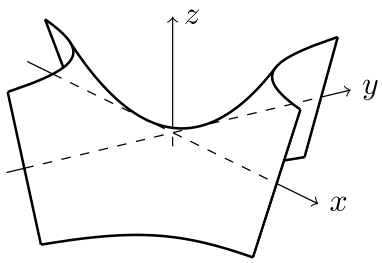
1. $f(x, y) = x^2 + y^2,\ D(f) = \mathbb{R}^2$.



2. $f(x, y) = \sqrt{1 - x^2 - y^2}$, $D(f) = \{(x, y) \mid x^2 + y^2 \le 1\}$.



3. $f(x, y) = xy,\ D(f) = \mathbb{R}^2$.



Occasionally the sort of extremum can also be determined by the circumstances — that is, one knows from the start that the function must have a maximum or must have a minimum.

Similarly to the case of one variable (6.5.e)) we can however draw on the second (partial) derivative for information about the character of an extremum. Let us assume here that these second partial derivatives, i.e. $f_{xx}, f_{yy}, f_{yx}, f_{xy}$, exist and are continuous. In particular, then, $f_{xy} = f_{yx}$ (23.4). Now the following state of affairs (given here without proof) holds:

- Necessary conditions:
  - (x_0, y_0) is an interior point of $D(f)$,
  - $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$,
  - the 2nd partial derivatives of f exist and are continuous.
- Notation: We define the number A as

$$A = A(x_0, y_0) = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 .$$

- Assertion:
  1. If $A > 0$, then f has a relative extremum at (x_0, y_0) and in fact specifically,
     - for $f_{xx}(x_0, y_0) < 0$ a relative maximum,
     - for $f_{xx}(x_0, y_0) > 0$ a relative minimum.
  2. If $A < 0$, then f has no relative extremum at (x_0, y_0).
  3. If $A = 0$, no conclusion can be drawn on this basis.

(23.7) Method of least squares

Method of least squares, continued:

Exercises: 1. Compute $\frac{\partial^2}{\partial x \partial y} \cos(x^2 + xy)$

2. In the differential equation

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial x^2}$$

partial derivatives appear. For this reason it is known as a *partial differential equation*. What condition do the numbers a and b need to satisfy, in order for $C(x,t) = \exp(ax+bt)$ to be a solution of the above equation?

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.