

This script is an extract, with gaps, from the book “Introduction to the Mathematical Treatment of the Natural Sciences I - Analysis” by Christoph Luchsinger and Hans Heiner Storrer, Birkhäuser Scripts. As a student you should buy the book as well and work your way through it completely during the course MAT 182. You are allowed to save this PDF and modify it as you like, for your own use during MAT 182. For further use outside of MAT 182, please contact the lecturer, Christoph Luchsinger of the University of Zürich, in advance. The copyright remains with Birkhäuser!

## **F. FUNCTIONS OF SEVERAL VARIABLES**

### **22. GENERAL DISCUSSION OF MULTIVARIATE FUNCTIONS**

(22.2) Introductory examples
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## (22.3) Definition of a multivariate function

By  $\mathbb{R}^2$  we denote the set of all *ordered pairs* of real numbers. “Ordered” in this context means that the 1st and 2nd position are considered as distinct:  $(1, 2)$  is not the same as  $(2, 1)$ . Geometrically we may understand  $\mathbb{R}^2$  as a plane by interpreting  $(x, y)$  as the point  $P$  whose cartesian coordinates are  $(x, y)$ .

Analogously,  $\mathbb{R}^n$  is the set of all ordered “ $n$ -tuples”

$$(x_1, x_2, \dots, x_n), \quad x_i \in \mathbb{R} \quad (i = 1, 2, \dots, n).$$

The set  $\mathbb{R}^n$  is also referred to as “ $n$ -dimensional space”. In particular, the 3-dimensional space  $\mathbb{R}^3$  consists of all ordered “triples”  $(x, y, z)$ .

With complete analogy to the univariate case (26.9) we state here the following general definition:

A *function of  $n$  variables* is given by assigning to every  $n$ -tuple  $(x_1, x_2, \dots, x_n)$  in a given (non-empty) subset  $D(f) \subset \mathbb{R}^n$  (the domain of  $f$ ) a uniquely determined real number  $f(x_1, x_2, \dots, x_n)$ .

It's clear that the investigation of a multivariate function will be more complicated than that of a function of one variable. However, most of the new phenomena can be observed already in the jump from one to two variables. This will allow us to illustrate the theory principally using examples which are *bivariate functions*.

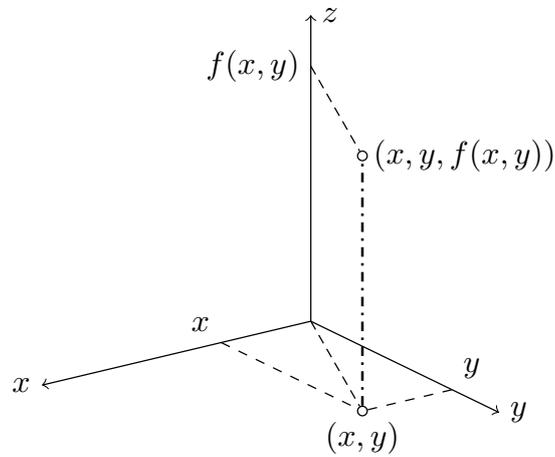
## (22.4) The graph of a bivariate function

Let  $f : D(f) \rightarrow \mathbb{R}$  (with  $D(f) \subset \mathbb{R}^2$ ) be a function of two variables. An ordered pair  $(x, y)$  in  $D(f)$  corresponds to a point in the  $x$ - $y$ -plane.

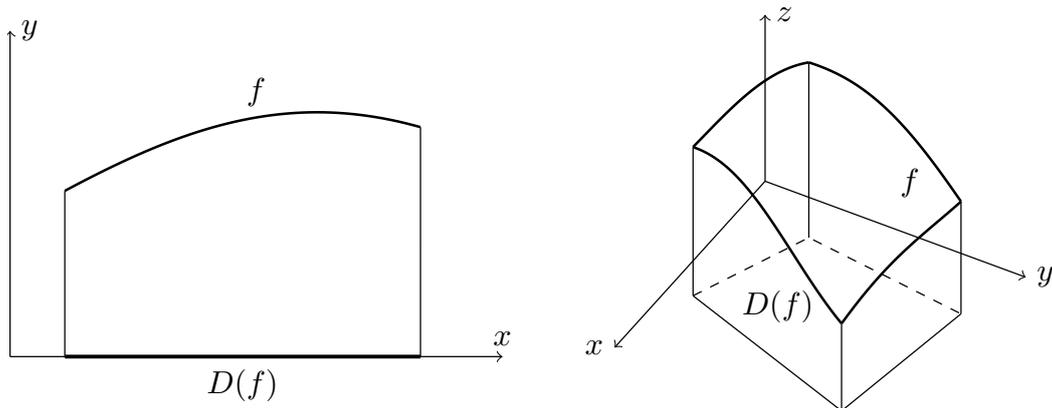
Now we consider a three-dimensional coordinate system and at every point  $(x, y) \in D(f)$  we mark the function's value  $f(x, y)$  in the direction of the  $z$ -axis. The set obtained in this way,

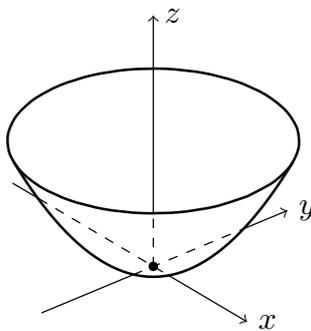
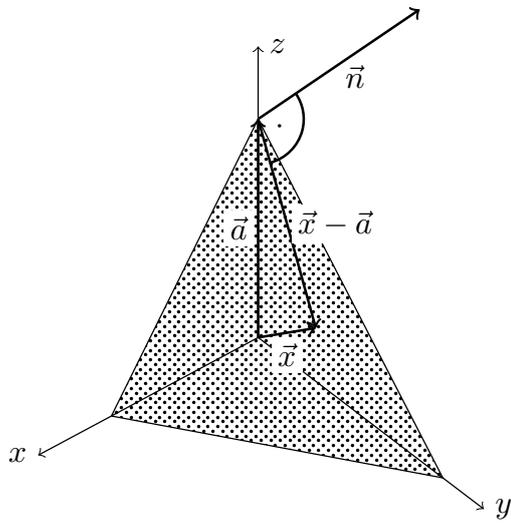
$$G = G(f) = \{(x, y, f(x, y)) \mid (x, y) \in D(f)\},$$

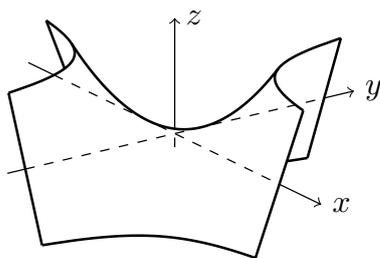
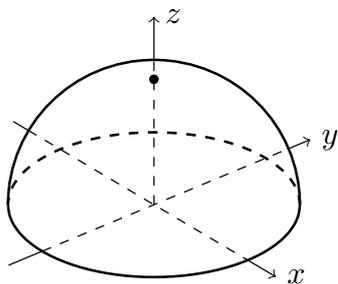
is called the graph of  $f$ .



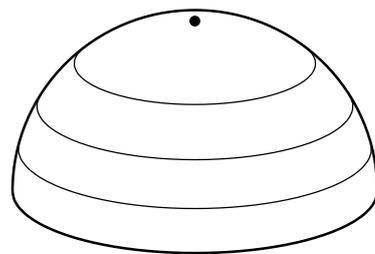
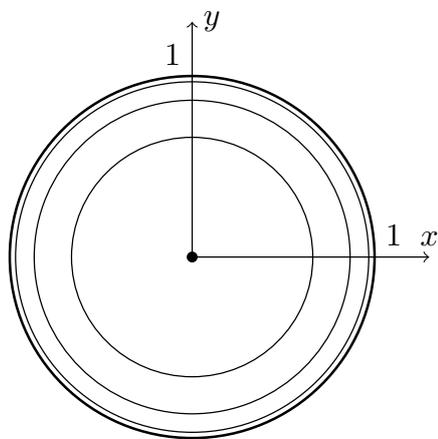
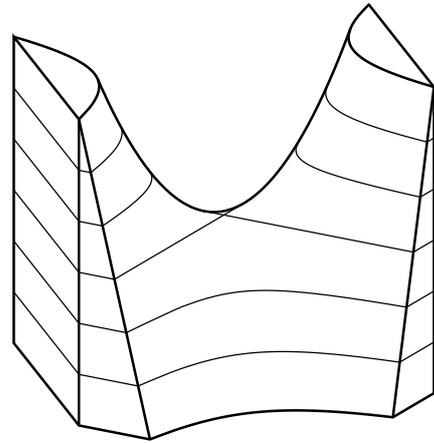
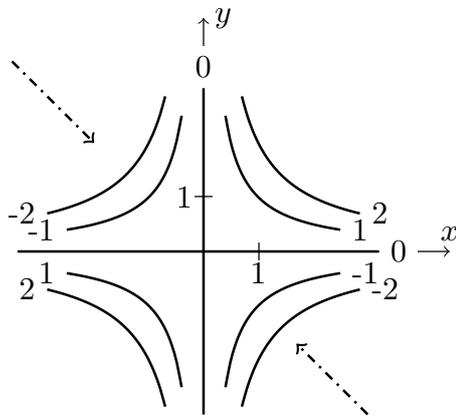
Instead of a graph as shown below left, we now have a “sheet” in three-dimensional space!



Examples I

Examples II

(22.5) Contour lines



## (22.6) Partial functions

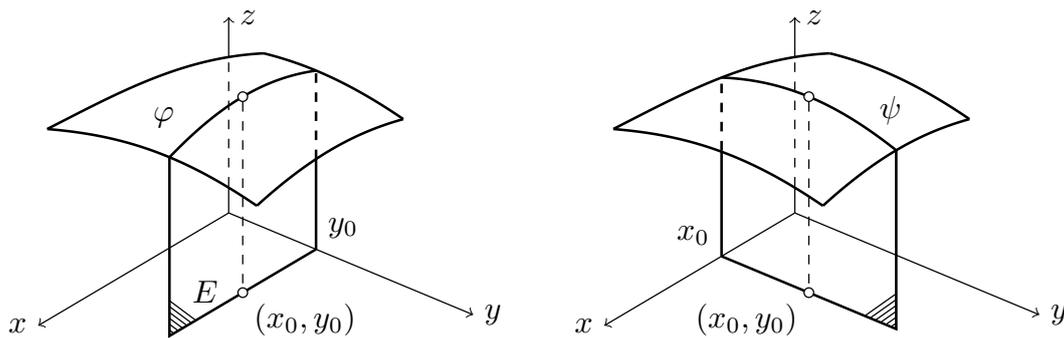
Following this example we consider the general case of a bivariate function

$$f : D(f) \rightarrow \mathbb{R}, \quad (x, y) \mapsto f(x, y), \quad D(f) \subset \mathbb{R}^2 .$$

Now we hold  $y$  constant ( $y = y_0$ ) and allow only  $x$  to vary. In this way we obtain a univariate function

$$x \mapsto f(x, y_0) ,$$

which is called a partial function of  $f$ .



In the following we will generally use  $\varphi$  to denote this function (more precise would be  $\varphi_{y_0}$ , with an explicit reference to  $y_0$ , but this notation is too cumbersome). Thus:

$$\varphi(x) = f(x, y_0) .$$

Correspondingly, we can define the partial function in the  $y$ -direction through  $x_0$ :

$$y \mapsto f(x_0, y) ,$$

where this time  $x$  is held constant ( $x = x_0$ ) and  $y$  is allowed to vary. In the following we will use  $\psi$  to denote this partial function:

$$\boxed{\psi(y) = f(x_0, y) .}$$

1. Let  $f(x, y) = \frac{1}{x^2 - y}$ . Sketch in all points in the  $x$ - $y$ -plane for which  $f$  is not defined. Draw the contour lines for the levels  $c = \pm\frac{1}{2}, \pm 1, \pm 2$ .

2. Let  $f(x, y) = x^2 + \frac{1}{y}$ , ( $y > 0$ ). Determine the partial functions for  $x = -1, 0, 1$  and for  $y = 1, 2, 3$ . Make a three-dimensional sketch.

**Important:**

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.