19. POWER SERIES

(19.2) Sequences (German: Folgen)

$\frac{1}{n}$

$\frac{2n^3+n^2+5n}{n^3}$

$1 + \frac{(-1)^n}{n}$

$(1 + \frac{1}{n})^n$

$(1 + \frac{x}{n})^n$

$(1 - \frac{1}{n})^n$

$\frac{2000n^3}{1.0001^n}$

The definitive ranking:

$(-1)^n$

$n(-1)^n$

And now the mathematically precise definitions:

Let (a_n) be a sequence. The number $a \in \mathbb{R}$ is said to be the *limit* (German: Grenzwert, Limes) of this sequence if the following condition is satisfied: For every number (no matter how small) $\varepsilon > 0$, there exists a natural number N (which depends on ε), such that:

$$a_n \in (a - \varepsilon, a + \varepsilon) \quad \text{for all} \quad n > N \,.$$

A sequence (c_n) converges improperly to ∞ (or diverges to ∞), if the following holds: for every number (no matter how large) C, there exists a natural number N with

$$c_n > C \quad \text{for all} \quad n > N \,.$$

Thus, in words: the successive terms eventually grow larger than any arbitrary number. We write then that

$$\lim_{n\to\infty} c_n = \infty \quad \text{or} \quad c_n \to \infty \quad \text{for} \quad n \to \infty \,.$$

The expression $\lim\limits_{n\to\infty} c_n = -\infty$ is defined correspondingly.

Notes:

(19.3) Series (German: Reihen)

To begin with: www.schweizermonat.ch/wo-mathematiker-an-grenzen-stossen

Closely related to the concept of a sequence is that of a series. You are already familiar with the geometric series. An example is

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots .$$

For the convergence of this series:

We say that the sum of this series is equal to 2. What we really mean by that is the following: we compute, in order,

$$\begin{aligned}
s_0 &= 1 && = 1\\
s_1 &= 1 + \tfrac{1}{2} && = 1.5\\
s_2 &= 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 && = 1.75\\
s_3 &= 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 && = 1.875\\
s_4 &= 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 && = 1.9375\,,
\end{aligned}$$

etc.

The sequence (s_n) thus constructed has the limit 2. Typically this is written as follows:

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots = 2\,.$$

The example above may be taken as a model for the general situation. Let

$$a_0, a_1, a_2, \dots$$

be a sequence of real numbers. From this we produce a new sequence

$$s_0, s_1, s_2, \ldots$$

called the *sequence of partial sums* of (a_k), by the construction

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ s_2 &= a_0 + a_1 + a_2 \\ &\vdots \\ s_n &= a_0 + a_1 + a_2 + \ldots + a_n = \sum_{k=0}^{n} a_k \, . \\ &\vdots \end{aligned}$$

Now if the sequence (s_n) converges and has the limit $s \in \mathbb{R}$, then we say that *the series* $\sum_{k=0}^{\infty} a_k$ *converges* (otherwise the series is said to *diverge*), and s is then said to be the *sum* of the series:

$$s = \sum_{k=0}^{\infty} a_k \, .$$

A series is thus nothing other than a sequence which is considered from a particular point of view.

Keep in mind that the symbol

$$\sum_{k=0}^{\infty} a_k$$

does not denote an infinite sum of real numbers (which would be impossible to define) but rather the limit of a certain sequence! For this reason the familiar rules for finite sums cannot be indiscriminately transferred to sequences. The following formulae are nevertheless valid, as can be shown:

$$\sum_{k=0}^{\infty}(a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k, \quad \sum_{k=0}^{\infty} c a_k = c \sum_{k=0}^{\infty} a_k, \quad (c \in \mathbb{R}) \, ,$$

insofar as the series on the right side are convergent.

(19.4) Examples of series

a) <u>The geometric series</u>

b) <u>The harmonic series</u>

(19.5) Power series

Why are we doing this?

1. It's beautiful mathematics: for instance, surprisingly it turns out that:

2. Functions such as $\sin(x)$, $\cos(x)$, e^x or even ln are heavy going. Polynomials are simple, for mental computation as well as for higher mathematics. The practical side of power series - even better than the linearisation by means of the differential which we saw in Chapter 7 - can be observed in the figure at the top of page 310 in Luchsinger/Storrer.

In this section we break new conceptual ground. We consider series whose summands are not simply numbers, but rather functions. In the case we discuss here, these functions will be power functions (e.g. $x \mapsto a_2(x-x_0)^2$, see below) and so we may speak of power series. We start by introducing the relevant terminology:

By a *power series centred at* x_0 we mean a series of the form

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots \,. \tag{1}$$

Here the a_k are to be thought of as fixed real numbers, while x is considered as a variable.

In practical application the power series we have to deal with will almost always be centred at $x_0 = 0$. Such a power series has the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \,. \tag{2}$$

However, the theory we set out here will always be for the general case.

If, in a power series, we substitute some number for x, the result will be an ordinary numerical series such as we discussed in (19.3). Here we consider once again the figure on page 295 in the book. For every sequence, then, the question of convergence reflexively presents itself. We will return to this question below; it is clear though that the answer will depend on the value of x. First we consider two examples:

Examples

1. If in (1) we set $x_0 = 0$ (we then find ourselves in case (2)) and $a_k = 1$ for all k, then we obtain

2. The series

$$1 + (x-1) + (x-1)^2 + (x-1)^3 + \dots$$

is also a geometric series, this time with ratio $x - 1$. Here we have a power series centred at 1. It converges for $|x-1| < 1$, or equivalently for $0 < x < 2$. For these values of x its sum is equal to

$$\frac{1}{1-(x-1)} = \frac{1}{2-x} \; . \qquad \boxtimes$$

Now we turn back to the general situation and consider the power series (1). As mentioned already, the very first question to ask about any series is whether it converges.

Since for a given power series the a_k are fixed, its convergence depends purely on x: for certain x (e.g. certainly for $x = x_0$, since here all the summands with the possible exception of a_0 are equal to zero) the power series will converge; in addition there may also exist values of x for which it diverges.

The *domain of convergence* D of a power series is the set of all $x \in \mathbb{R}$ for which the series converges. This D is of course dependent on the power series being considered, i.e. on the coefficients a_k. For every x in D the sum of the series is a real number. Thus the correspondence

$$p : D \to \mathbb{R} \; , \quad x \mapsto p(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

defines a function p. This *power series function* associates to every x — that is, to every x for which the sum of the power series is even defined — the value of this sum.

Let us revisit the two examples above in this context. In Example 1, $D = \{x \in \mathbb{R} \mid -1 < x < 1\} = (-1, 1)$ and for $x \in D$ we have

$$p(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} x^k \; .$$

In Example 2, $D = \{x \in \mathbb{R} \mid 0 < x < 2\} = (0, 2)$ and for $x \in D$

$$p(x) = \frac{1}{2-x} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \ldots = \sum_{k=0}^{\infty} (x-1)^k \ .$$

We return to the consideration of power series functions in general. The relationship

$$p(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

means that $p(x)$ is the limit of the partial sums s_n of

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots \ .$$

Here the n-th partial sum is

$$s_n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n \ ,$$

as for instance in Example 2

$$s_n = 1 + (x-1) + \ldots + (x-1)^n \ ,$$

simply a polynomial. Thus, a power series function is a limit of polynomial functions, which we now would like to denote by $p_n(x)$ rather than by s_n. Since $p_n(x) \to p(x)$, the function $p(x)$ can be approximated to any desired precision by computing the polynomial $p_n(x)$ for a sufficiently large n (notice that $p_n(x)$ is obtained simply by "truncating" the power series for $p(x)$). In this way, the values of power series functions can be determined with arbitrary precision.

We look at a few more concrete examples, starting with the geometric series. As we know,

(3) $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad \text{for} \quad |x| < 1 \ .$$

We can also interpret this formula conversely, as stating that we have found a way to represent the function $\frac{1}{1-x}$ by a power series. We can also say that we have found a *power series expansion* of the function.

It will be our goal to develop power series representations for a few of the most important functions (such as e^x, $\sin x$, etc.) In (19.8) we will see that the following holds:

(4) $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \text{for all } x \in \mathbb{R} \ .$$

Let's play around with some calculations:

(19.6) Arithmetic rules for power series - what's allowed and what isn't:

We consider a power series

$$p(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

with domain of convergence D.

a) Power series with the same centre x_0 may be added and subtracted termwise: if

$$q(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

is another power series, then

$$p(x) + q(x) = \sum_{k=0}^{\infty} (a_k + b_k)(x - x_0)^k ,$$

for all x for which both $p(x)$ and $q(x)$ converge. An analogous formula holds for the difference.

b) Power series may be multiplied termwise by a constant: for every real number c and for all $x \in D$,

$$cp(x) = \sum_{k=0}^{\infty} c a_k (x - x_0)^k .$$

Formulae a) and b) follow from the general rules for series (at the end of (19.3)).

Example

The next two rules are important for the infinitesimal calculus:

c) A power series may (in D) be differentiated termwise: from

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots$$

it follows that

$$p'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + 4a_4(x-x_0)^3 + \dots \; .$$

d) A power series may (in D) be integrated termwise (the limits of integration a, b must lie in D): from

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots$$

it follows that

$$\begin{aligned}\int_a^b p(x)\,dx &= \int_a^b a_0\,dx + \int_a^b a_1(x-x_0)\,dx + \int_a^b a_2(x-x_0)^2\,dx + \dots \\ &= a_0 x\Big|_a^b + \frac{1}{2}a_1(x-x_0)^2\Big|_a^b + \frac{1}{3}a_2(x-x_0)^3\Big|_a^b + \dots \; .\end{aligned}$$

<u>Examples</u>

1. As a first application, we differentiate the relation (3)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

on both sides and obtain (with rule c)) the new power series

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots \quad \text{for } |x| < 1 \; . \tag{8}$$

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2. Next we integrate the series (7) from 0 to t ($t \in D = (-1,1)$):

$$\int_0^t \frac{1}{1+x^2}\,dx = \int_0^t 1\,dx - \int_0^t x^2\,dx + \int_0^t x^4\,dx - \int_0^t x^6\,dx + \dots \; .$$

After a short computation ($\arctan x$ is an antiderivative of $1/(1+x^2)$) we find:

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots \quad \text{for } |t| < 1 \; . \tag{9}$$

It is possible to show that this formula also holds for $t = 1$, which since $\arctan 1 = \frac{\pi}{4}$ yields the following surprising relation:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \; .$$

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3. We have already mentioned repeatedly the fact that the function $f(x) = e^{-x^2}$ has no elementary antiderivatives. In particular, the integral

$$\int_0^x e^{-t^2}\,dt$$

cannot be represented in closed form by elementary functions. However...

(19.7) Taylor series

Now we set ourselves the task of constructing a power series expansion of a given function $f(x)$ (e.g. e^x or $\sin x$). To this end, we start by assuming that the function $f(x)$ has a power series expansion with centre x_0:

$$(*) \quad f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \ldots + a_k(x - x_0)^k + \ldots .$$

This means, then, that the power series on the right converges for all x in a certain neighbourhood of x_0 and has $f(x)$ as its sum.

Our immediate task is thus to determine the coefficients a_k:

In the general case,

$$f^{(k)}(x_0) = k(k-1)(k-2)\dots 3\cdot 2\cdot a_k = k!a_k$$

and with this we have found the desired formula for a_n, namely

$$\boxed{a_k = \frac{f^{(k)}(x_0)}{k!}, \quad k = 0, 1, 2, \dots}$$

Here as usual $k! = k(k-1)(k-2)\dots 3\cdot 2\cdot 1$, (see (26.4.a)). The formula is also valid for $k = 0$ because of the conventions $0! = 1$ (26.4) and $f^{(0)}(x) = f(x)$ (4.5).

After all these computations, it is not a bad idea if we take a moment to review precisely what it is we have been doing. We have shown the following: *if* the function $f(x)$ can be represented by a power series at all in a neighbourhood of x_0, then the coefficients *must* be those given by the above formula; the series therefore has the form

$$\boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots .}$$

This power series is called the *Taylor series of the function f with centre x_0*. For $x_0 = 0$ this is sometimes also referred to as the *Maclaurin series* of the function.

The question of the domain of convergence D remains open for the moment, and must be investigated separately, you can read more (though not a complete account) in Luchinger/Storrer. On the basis of the formula's derivation, we naturally hope that this series will converge to $f(x)$ for all $x \in D$.

In summary, for a given function $f(x)$, there are two questions we need to ask:

- What is the domain of convergence D of the Taylor series?
- In that domain, does it really converge *to* $f(x)$?

Only when the answer to the second question is "yes" may we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{for} \quad x \in D .$$

Answering these two questions requires what is often a rather intricate investigation, which we will not be able to pursue here. In the next section we will therefore state such results without proving them.

(19.8) Computation of Taylor series

Now we will determine the Taylor series of several important functions, in each case centred at 0. For this we need all the derivatives of these functions at the point $x_0 = 0$.

a) Exponential function $f(x) = e^x$

b) <u>Sine function</u> $f(x) = \sin x$

c) <u>Cosine function</u> $f(x) = \cos x$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\ , \quad x \in \mathbb{R}\ .}$$

(Since $\cos(-x) = \cos x$, only even powers appear here.) Compactly formulated:

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}\ , \quad x \in \mathbb{R}\ .$$

d) Logarithm function $f(x) = \ln(1+x)$

As $\ln x$ is not defined at the point $x = 0$, we will consider, not $\ln x$, but rather $\ln(1+x)$.

(19.9) Taylor polynomials

As we have just seen, most of the functions which are important to us here can be represented as a sum of their (infinite) Taylor series. If we terminate this sequence after the n-th term, we obtain in this way a polynomial, namely

$$\begin{aligned} p_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \\ &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \ . \end{aligned}$$

This polynomial is said to be the *Taylor polynomial of degree* n (with *centre* x_0) of the function f. Note once again that x_0 is a fixed number, only x is a variable.

What is the connection between the function f and its n-th Taylor polynomial $p_n(x)$? This polynomial is nothing other than a partial sum of the Taylor series. Thus,

$$p_n(x) \to f(x) \quad \text{for} \quad n \to \infty \ ,$$

for all x for which the Taylor series of f actually converges to f (that is, for $f(x) = e^x$ for all $x \in \mathbb{R}$, for $f(x) = \ln(1+x)$ for all $x \in (-1, 1]$, cf. (19.8)). This means that $f(x)$ will be better approximated by the (easily computed!) polynomial $p_n(x)$, the larger n is, see also the numeric examples further below.

Furthermore (see Luchsinger/Storrer, page 308) it holds that:

$$p_n^{(k)}(x_0) = f^{(k)}(x_0) \quad \text{for } k = 0, 1, \ldots, n \ .$$

In words: At the point $x = x_0$, the functions p_n and f have matching values — of the functions themselves, and also of their first n derivatives. It follows that the approximation of f by p_n is a good one, above all in the immediate region of x_0,

$$p_n(x) \approx f(x) \quad \text{near } x_0 \ ,$$

and in fact the approximation is better, the larger n is, as we have already seen above. Considered geometrically, the graph of $p_n(x)$ lies closer to that of $f(x)$, the larger n is, cf. Example d) below.

We note here that to form the Taylor polynomial $p_n(x)$, only the existence of the first n derivatives of $f(x)$ is necessary (and not the existence of the Taylor series).

The Taylor polynomial of degree 1 is given by

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0) \ .$$

This is nothing other than the best approximation of $f(x)$ by a linear function (near x_0), which was already dealt with in (7.3). A comparison with the formula for the Taylor series shows that this linear approximation is obtained by simply discarding the terms "of higher order in Δx" ($\Delta x = (x - x_0)$), i.e. the squares, cubes, ... of $(x - x_0)$.

Examples and observations

From the formulae derived in (19.8) for the Taylor series, we may immediately read off formulae for the Taylor polynomials:

a) $f(x) = e^x$.

$$p_1(x) = 1 + x,$$
$$p_2(x) = 1 + x + \frac{x^2}{2},$$
$$p_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad \text{etc.}$$

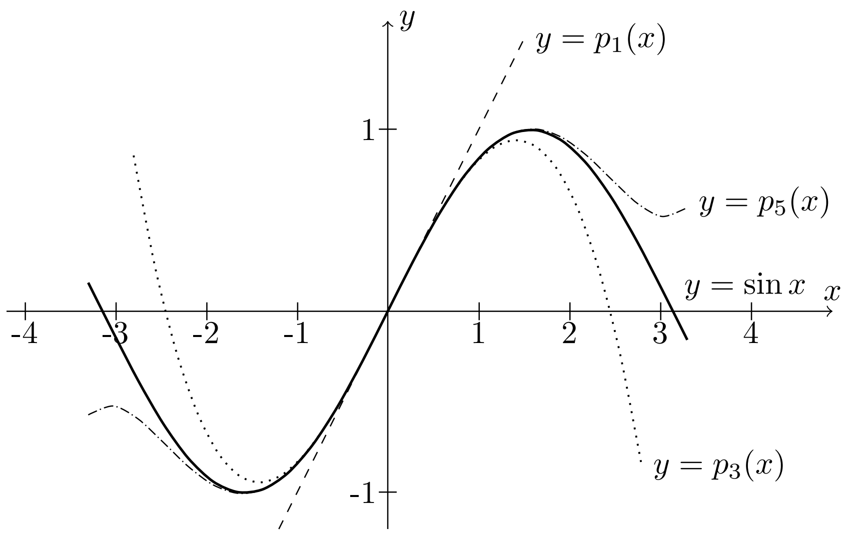
c) As already mentioned, these Taylor polynomials are approximations of $f(x)$. The table below contains some numerical values to compare f and p_3.

$f(x)$	$p_3(x)$	$f(0.2)$	$p_3(0.2)$
e^x	$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$	1.2214	1.2213
$\sin x$	$x - \frac{x^3}{6}$	0.1987	0.1987
$\cos x$	$1 - \frac{x^2}{2}$	0.9801	0.9800
$\ln(1 + x)$	$x - \frac{x^2}{2} + \frac{x^3}{3}$	0.1823	0.1827

d) In the following figure you can compare the graph of the function $y = \sin x$ with those of the Taylor polynomials

$p_1(x) = x,$
$p_3(x) = x - \frac{x^3}{6}\,,$
$p_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}.$

As you can see, in the region near 0 the approximation is very good. At greater distances from the origin, the approximation through $p_5(x)$ is the best one, which on the strength of our derivation of the Taylor polynomial is unsurprising.



Finally, a couple of exercises:

1. Compute the sum of the following geometric series:

$$1 - x^3 + x^6 - x^9 + x^{12} - \dots \ , \quad |x| < 1$$

2. Verify the familiar formulae for the derivatives of the functions e^x, $\sin x$, $\cos x$ with the help of their Taylor polynomials.

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.