This script is an extract, with gaps, from the book “Introduction to the Mathematical Treatment of the Natural Sciences I - Analysis” by Christoph Luchsinger and Hans Heiner Storrer, Birkhäuser Scripts. As a student you should buy the book as well and work your way through it completely during the course MAT 182. You are allowed to save this PDF and modify it as you like, for your own use during MAT 182. For further use outside of MAT 182, please contact the lecturer, Christoph Luchsinger of the University of Zürich, in advance. The copyright remains with Birkhäuser!

E. DEVELOPMENT OF INFINITESIMAL CALCULUS

17. INVERSE FUNCTIONS (UMKEHRFUNKTIONEN)

(17.2) Inverse trigonometric functions

a) Introduction

It often happens that we have the value of a trigonometric function, e.g. the sine function, and need to calculate the corresponding angle (here always measured in radians). To put it differently: that in the equation

$$y = \sin x \tag{1}$$

y is given and x is to be determined, i.e. we need to solve this equation for x.

To summarise it in words:

$$\arcsin y \text{ is the number in } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \text{ whose sine is } = y.$$

For every y in $[-1, 1]$ there is a uniquely determined number $\arcsin y$, and so the mapping

$$y \mapsto \arcsin y$$

serves to define a new function, the arcsine function, or arcsine for short. Its domain of definition is the interval $[-1, 1]$. Completely analogously, we obtain from the remaining trigonometric functions the new functions arccosine, arctangent, and arccotangent (cf. (17.2.c) through (17.2.e)). These are referred to collectively as the *inverse trigonometric functions* (German: zyklometrische Funktionen, Arcus-Funktionen) Here we will further investigate some properties of the arcsine.

By definition the expressions $y = \sin x$ and $x = \arcsin y$ mean one and the same thing for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $y \in [-1, 1]$. Expressed algebraically:

$$(2) \qquad y = \sin x \Longleftrightarrow x = \arcsin y \; .$$

In such a situation one speaks of *inverse functions* (cf. (17.3)) (German: Umkehrfunktionen). One says that the arcsine is the inverse function of the sine and vice versa.

What happens when we take the composition of two functions which are inverses of each other? If in the formula $y = \sin x$ we replace the number x with $\arcsin y$ – which, based on (2), we can do – then we obtain

$$(3) \qquad y = \sin(\arcsin y), \quad (y \in [-1, 1]) \; .$$

In the same way, we can replace y in $x = \arcsin y$ with the value $\sin x$ and find

$$(4) \qquad x = \arcsin(\sin x), \quad (x \in [-\frac{\pi}{2}, \frac{\pi}{2}]) \; .$$

We conclude this discussion by constructing the graph of the arcsine function. We start with the relation

$$(2) \qquad y = \sin x \Longleftrightarrow x = \arcsin y \quad \text{for } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \; .$$

The graph of the function $y = \sin x$ (when its domain is limited to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$) consists of all ordered pairs (x, y) with $y = \sin x$, $(x \in [-\frac{\pi}{2}, \frac{\pi}{2}])$. Based on (2), these are exactly the same ordered pairs which also describe the relation $x = \arcsin y$. It follows that $y = \sin x$ and $x = \arcsin y$ have the same graph. However, for $x = \arcsin y$ the independent variable is not x as usual, but rather y which is unfamiliar. To get back to the familiar setting, then, we need to exchange x and y, which corresponds to a reflection across the line $y = x$.

So to obtain the graph of the arcsine function, we restrict the sine function to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and then reflect this graph across the line $y = x$:

The graphs and properties of the other inverse trigonometric functions are derived completely similarly; please read the remaining parts of (17.2) after this lecture.

A few points about calculators (German: Taschenrechner, TR) (no guarantees):

calculators in the exercises, YES; in the exam, NO!

* The button $\sin^{-1}$ is normally the inverse function ("arcsin") and not

the reciprocal $\frac{1}{\sin(x)}$. Please test this yourself **in advance**!

* In general, f^{-1} is the inverse function and

$$(f(x))^{-1} = \frac{1}{f(x)}$$

the reciprocal at the point x.

* Check the settings for degrees and radians on your own calculator. As a check: in degrees, $\sin(30°) = 0.5$ and correspondingly in radians $\sin(\pi/6) = 0.5$, since $30° \mathrel{\hat{=}} \pi/6$.

(17.3) Inverse functions

For completely detailed definitions and information, please read the corresponding section in the textbook. Here we will discuss:

a) What are we trying to do and what can go wrong?
b) How can we restrict ourselves so that it doesn't go wrong?
c) What does that enable us to do?
d) Notable examples

a) What are we trying to do and what can go wrong?

* $y = f(x)$, now we have y and want to find x.

* Domains of definition have to match: $10 = \sin(x)$.

* For a given y there might be multiple x where $f(x) = y$ is satisfiederfüllt :

* From $y = x^2 = 1$ we obtain $x_1 = -1$ and $x_2 = 1$.

* NB: both statements are true: $\sqrt{9} = 3$ and the equation $x^2 = 9$ has the solution set $L = \{-3, 3\}$. These are simply two different questions and so they have two different answers!

b) How can we restrict ourselves so that it doesn't go wrong?

* Check domains of definition: usually this is obvious (except for badly written code)

* A related problem with trigonometric functions: don't confuse degrees and radians.

* Solution: Restrict the domain of $f : D(f)$ in such a way that f is **injective, i.e. there is only 1 value possible "going backwards"**

* Definition: f is injective if: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. For example this is always fulfilled by strictly monotonic functions.

c) What does that enable us to do?

d) Notable examples

(17.4) Derivative of the inverse function

We start by repeating the derivatives of sine, cosine, and friends and their inverse functions. Some of these were simply stated without justification in earlier chapters; we are about to understand why they look as they do.

* From high school: $(\sin(x))' = \cos(x)$ and $(\cos(x))' = -\sin(x)$

* Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\cot(x) = \frac{\cos(x)}{\sin(x)}$, by using the quotient rule (we may have done these in the weekly exercises, if not go ahead and try them at home) we get:

$$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

and

$$(\cot(x))' = -\frac{1}{\sin^2(x)} = -(1 + \cot^2(x)).$$

And – crunch time now – how about the derivatives of their inverses?

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}},$$

$$(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}},$$

$$(\arctan(x))' = \frac{1}{1+x^2},$$

$$(\text{arccot}(x))' = -\frac{1}{1+x^2}.$$

In practice these last 4 equations turn out to be useful above all in the opposite direction: if you need to *integrate* $\frac{1}{1+x^2}$, now you have an antiderivative for it (read the third equation from right to left). Such expressions do come up in real-world settings! We will just prove the first one as an example (p +3).

A graph to illustrate the derivation of the formula

Examples I:

Examples II:

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.