16. SELECTED METHODS FOR SOLVING

(16.2) Initial conditions (Anfangsbedingungen)

(16.3) Slope field (Richtungsfeld)

Again we consider the general differential equation

$$y' = F(x, y) \ .$$

and in addition, we let $y = f(x)$ be a solution of this equation.

We want to investigate the graph of f. To this end we consider the point (x_0, y_0) with $y_0 = f(x_0)$, which of course lies on the graph of f. But f is a solution of the above differential equation; thus

$$f'(x_0) = F(x_0, y_0) \ .$$

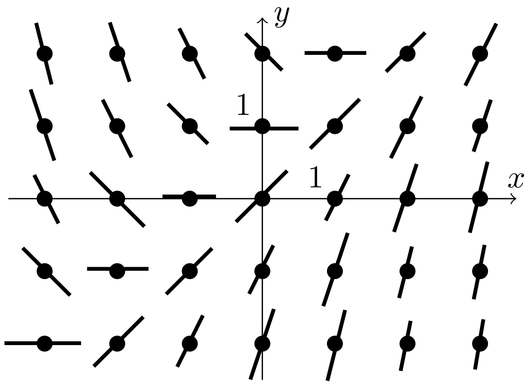
The number $f'(x_0)$ is exactly the slope of the graph of f (more precisely: the tangent to the graph of f) at the point (x_0, y_0). In other words: the solution curve which passes through (x_0, y_0) has at this point the slope $F(x_0, y_0)$. The significance of this is that, since $F(x, y)$ is given, we now have a way to compute that slope without knowing the solution function $f(x)$.

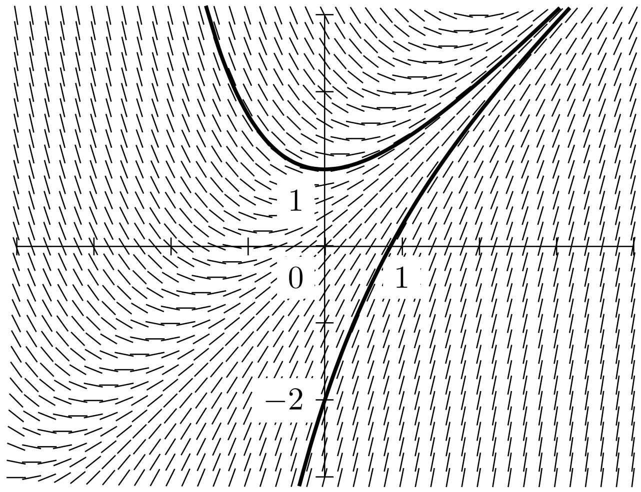
Example

Let $y' = x - y + 1$. (In (16.6.1) we will solve this equation algebraically). To begin with, we tabulate y' as it depends on x and y.

y \ x	-3	-2	-1	0	1	2	3
2	-4	-3	-2	-1	0	1	2
1	-3	-2	-1	0	1	2	3
0	-2	-1	0	1	2	3	4
-1	-1	0	1	2	3	4	5
-2	0	1	2	3	4	5	6

Next, through each point (x, y) in the plane we can draw a short line segment with the corresponding slope $y' = F(x, y) = x - y + 1$. At the point $(1, 1)$ the segment has the slope $F(1, 1) = 1$, at the point $(0, 2)$ the slope $F(0, 2) = -1$ and so on. We obtain the following hand-drawn sketch. Naturally one can also delegate the computation and drawing to a computer, to get more comprehensive images like the one on the right.





⊠

The object represented by both sketches is called the *slope field* of the differential equation. If (as in the second sketch) we include sufficiently many "direction elements", the approximate graphs of solution curves can be recognised. The two curves drawn above for example are particular solutions of the differential equation. As the upper curve passes through the point $(0, 1)$, it must be the particular solution corresponding to the initial condition $(x_0, y_0) = (0, 1)$; similarly the lower curve for example corresponds to the initial condition $(x_0, y_0) = (0, -2)$. In general we can see here that for every point in the plane, there is a solution curve which passes through it.

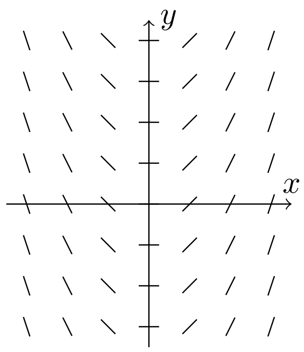
The development of this solution curve can be visualised as follows: we begin at (x_0, y_0), proceed in the direction $(1, y')$ $(y' = F(x_0, y_0))$ a little bit further to a point (x_1, y_1), where we compute a new slope $F(x_1, y_1)$, proceed further in the direction $(1, y')$ (with a new $y' = F(x_1, y_1)$) to some point (x_2, y_2), and so on. In this way we obtain a chain of line segments; the smaller the steps, the more closely this chain approximates the solution curve through (x_0, y_0). In the limit ("infinitely small" steps, take with a pinch of salt) the desired solution curve emerges (cf. (21.4)).

Looking again at this slope field, we find ourselves starting to suspect that the function $y = x$ might be a particular solution of the differential equation $y' = x - y + 1$. In fact, this can be immediately verified algebraically: we get $y' = 1$ and $x - y + 1 = x - x + 1 = 1$, and so for all x the equation $y' = x - y + 1$ is satisfied.

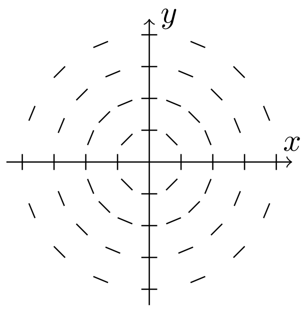
Two more examples follow. In these simple cases the solutions of the differential equation can be guessed by looking at the slope field:

- In the first example the solution curves are parabolas (the general solution is in fact $y = \frac{1}{2}x^2 + C$).
- In the second example the solution curves are circles centred on the origin (cf. the

mathematical solution in (16.10.3)).



$y' = x$



$y' = -\dfrac{x}{y}$

It should be mentioned in closing that these slope fields are not to be confused with the vector fields of (14.3).

(16.4) Linear differential equations of the 1st order

Now we come to the first systematic method of solving. This applies to so-called "linear differential equations of the 1st order".

A differential equation of the 1st order is called *linear* if it has the form

$$y' = p(x)y + q(x)$$

where p and q are functions of x.

The designation "linear" refers to the fact that y (and y') only appear to the power 1. The functions $p(x)$ and $q(x)$, on the contrary, do not need to be linear at all.

We may divide linear differential equations into two cases:

A linear differential equation is called *homogeneous* if $q(x) = 0$; otherwise we say it is *inhomogeneous*.

If

$$y' = p(x)y + q(x)$$

is a linear differential equation, then

$$y' = p(x)y$$

is called the *associated homogeneous equation*. For example, $y' = xy + 1$ is an inhomogeneous linear differential equation (of the 1st order) and $y' = xy$ is the associated homogeneous equation.

(16.5) Solution method for linear differential equations of the 1st order

a) <u>Overview</u>

To solve a linear differential equation, we first solve the associated homogeneous equation and then afterward employ the "method of variation of constants" to find the solutions of the original equation. We'll go through this now in detail.

b) <u>Solution of the homogeneous equation</u>

Let

$$y' = p(x)y$$

be a homogeneous linear differential equation. The solution can be guessed: let $P(x)$ be any antiderivative of $p(x)$ (i.e. $P'(x) = p(x)$). Then

$$\boxed{y = Ke^{P(x)}, \quad K \text{ an arbitrary constant}}$$

is the general solution of the homogeneous equation, since when we take the derivative the chain rule yields

$$y' = Ke^{P(x)} \cdot P'(x) = Ke^{P(x)} \cdot p(x) = p(x)y \ .$$

It should also be noted that the constant function $y = 0$ is always a solution of any homogeneous linear differential equation; set $K = 0$ and check by calculating:

Why not $P(x) + C$, i.e. $Ke^{P(x)+C}$?

<u>Examples</u>

1. $y' = y \sin x$: $p(x) = \sin x$, $P(x) = -\cos x$; solution: $y = Ke^{-\cos x}$; check by calculating:

2. $y' = e^x y$, $x > 0$.

c) <u>Solution of the inhomogeneous equation ("variation of constants")</u>

We consider once again the inhomogeneous linear differential equation

$$y' = p(x)y + q(x) \tag{1}$$

and the associated homogeneous equation

$$y' = p(x)y \ . \tag{2}$$

As just discussed, (2) has the general solution

$$y = Ke^{P(x)} \ , \tag{3}$$

where $P(x)$ is an antiderivative of $p(x)$.

Now we apply the method of *"variation of constants"* . This method consists of *assuming* that the solution of the inhomogeneous equation (1) has the form

$$\boxed{y = K(x)e^{P(x)}} \ . \tag{4}$$

In other words, we replace the constant K in (3) with some (for the moment still undetermined) function $K(x)$, and then try to determine this function in such a way that (4) is a solution of (1).

To find this function $K(x)$, we take the derivative of y (product rule and chain rule!):

$$y' = K'(x)e^{P(x)} + K(x)p(x)e^{P(x)} \ . \tag{5}$$

Here we have also used the fact that $P'(x) = p(x)$. Now we substitute y from (4) and y' from (5) into the original equation

$$y' = p(x)y + q(x) \ .$$

We obtain

$$K'(x)e^{P(x)} + \underline{K(x)p(x)e^{P(x)}} = \underline{p(x)K(x)e^{P(x)}} + q(x) \ .$$

The underlined terms add to zero (which was the point of this approach) and all that remains is

$$K'(x)e^{P(x)} = q(x) \ .$$

By multiplying with the reciprocal $e^{-P(x)}$ of $e^{P(x)}$ we obtain

$$K'(x) = q(x)e^{-P(x)} \ ,$$

from which we can compute $K(x)$ as an antiderivative of $q(x)e^{-P(x)}$:

$$K(x) = \int q(x)e^{-P(x)}\,dx + C\ .$$

Finally, we substitute the expression thus obtained for $K(x)$ into the initial assumption (4) and find as a solution of the inhomogeneous linear differential equation

$$y = \left(\int q(x)e^{-P(x)}\,dx + C\right) e^{P(x)}\ ,$$

where $P(x)$ is an arbitrary antiderivative of $p(x)$. Formulated differently:

$$y = (K_0(x) + C)e^{P(x)}\ , \tag{6}$$

where $K_0(x)$ is some particular, arbitrarily chosen antiderivative of $q(x)e^{-P(x)}$.

The method for solving an inhomogeneous linear differential equation

$$y' = p(x)y + q(x)$$

can be summarised then as follows:

1) Solve the associated homogeneous differential equation

$$y' = p(x)y\ .$$

The solution has the form

$$y = Ke^{P(x)}\ ,$$

where $P(x)$ is an antiderivative of $p(x)$.

2) Vary the constants K, i.e. make the assumption

$$y = K(x)e^{P(x)}\ ,$$

where the function $K(x)$ still has to be determined.

3) If you apply this assumption to the original inhomogeneous differential equation, after some rearrangement you get an expression for $K'(x)$, from which you can try to compute $K(x)$.

(16.6) Examples of the solution method - other examples than in the textbook

1. $y' = -y + e^x$

2. $y' = \frac{y}{x} + 1 \quad (x > 0)$

Aufgabe 16-3d); $y' = \frac{3}{x}y + x, x > 0$

(16.9) Separation of Variables

We proceed now to another systematic method of solution, the method of *separation of variables*. This method can always be applied if the differential equation has the form

$$\boxed{y' = r(x)s(y)}$$

i.e. when the right side $F(x, y)$ can be written as the product of two functions of one variable (one function of x and one function of y). As a special case the functions $r(x)$ resp. $s(y)$ may also be constant functions with the value 1, which means that the differential equations

$$y' = r(x) \quad \text{and} \quad y' = s(y)$$

are also subject to this method.

We illustrate the method by means of a simple, familiar example; $y' = \alpha y$, where $y > 0$ and $t > 0$; $\alpha \in \mathbb{R}$:

In this situation one proceeds according to the following method:

1) Write the differential equation in the form

$$\frac{dy}{dx} = r(x)s(y) \ .$$

2) Bring all terms with y onto the left side, all terms with x to the right side. In the process, we should formally multiply by dx:

$$\frac{dy}{s(y)} = r(x)\, dx \ .$$

3) Take the indefinite integral of both sides:

$$\int \frac{dy}{s(y)} = \int r(x)\, dx + C \ .$$

Do not forget the constant of integration C!*

This gives you the equation

$$S(y) = R(x) + C \ ,$$

where $S(y)$ is an antiderivative of $\frac{1}{s(y)}$, $R(x)$ an antiderivative of $r(x)$. Through this equation, y is given "implicitly" as a function of x. This y is then the general solution of the differential equation.

4) Solve for y (if possible).

5) Check whether there are constant solutions which were not already included.

Comment on 5); find constant solutions of $y' = x(y^2 - 4)$.

* Properly speaking, each integral gives rise to such a constant; we combine them into one though.

(16.10) Examples of separation of variables

1. The linear homogeneous differential equation

We have already seen in (16.5.b) that the differential equation

$$y' = p(x)y$$

has the general solution

$$y = Ke^{P(x)}$$

where $P(x)$ is an antiderivative of $p(x)$.

This equation can also be solved using separation of variables. We demonstrate this possibility with a concrete example, namely the equation

$$y' = e^x y\ ,$$

and go through the five steps of the "recipe" in (16.9) in order.

2. <u>The linear differential equation with constant coefficients</u>

The differential equation

$$y' = ay + b, \ a, b \in \mathbb{R}, \quad a \neq 0$$

was already solved in the textbook (not the lecture) in (16.8). Our new method proves itself here as well though. We write

$$y' = a(y + \frac{b}{a})$$

(i.e. we take $r(x) = a, \ s(y) = y + \frac{b}{a}$).

3. $y' = -\dfrac{x}{y}$ (slope field in (16.3)).

4. $y' = e^{x-y}$ (not in the textbook)

Terminology: in *this* course we use the following: a **particular** solution (German: spezielle Lösung) is a solution which satisfies given initial conditions (in contrast to the **general** solution (German: allgemeine Lösung)). **Constant (=stationary)** solutions are solutions which do not vary over time. **Singular** solutions are those which are not captured as part of the general solution using the method of separation of variables - these are also constant solutions (example in the textbook: (16.10.4)).

You can always check proposed solutions:

if you think you have found a solution of a differential equation:

substitute it into the DE to check!

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.