14. INTEGRATION OF VECTOR FUNCTIONS

(14.2) Ordinary integration of vector functions

If a vector function $\vec{r}(t)$ is given by

$$\vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix},$$

then as we know (cf. (8.5)) the derivative is taken coordinate-wise:

$$\dot{\vec{r}} = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix}.$$

Completely analogously, we can compute the definite integral coordinate-wise. We define:

$$\boxed{\int_a^b \vec{r}(t)\,dt := \begin{pmatrix} \int_a^b x_1(t)\,dt \\ \int_a^b x_2(t)\,dt \\ \int_a^b x_3(t)\,dt \end{pmatrix}.}$$

The value of this integral is another vector. Let's consider some applications of this concept.

<u>Examples</u>

As a preliminary, we repeat the relationship between velocity and distance which we found in Chapter 9 in the one-dimensional case.

We have:

1. A point particle moves in such a way that its velocity vector

$$\vec{v}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix}$$

at any time t is known. At time t_0 its location is that given by the position vector $\vec{r}(t_0) = \vec{r}_0$. What is its location at time t?

Let $\vec{r}(t)$ denote the position vector of the particle at time t. Following (8.4) we see that $\vec{v}(t) = \dot{\vec{r}}(t)$. For the first coordinate function in particular, it holds that

$$v_1(t) = \dot{x}_1(t) .$$

By integrating we obtain

$$\int_{t_0}^{t} v_1(u)\, du = \int_{t_0}^{t} \dot{x}_1(u)\, du = x_1(t) - x_1(t_0) ,$$

since of course $x_1(t)$ is an antiderivative of $\dot{x}_1(t)$. (Because t is already taken for the upper limit of integration, we have changed the notation to make u the variable of integration.) Analogous formulas hold for the two remaining coordinates. These three relationships can be summarised in accordance with the above definition into the following vector equation::

$$\int_{t_0}^{t} \vec{v}(u)\, du = \vec{r}(t) - \vec{r}(t_0) .$$

To our initial question, then, we obtain the answer

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(u)\, du . \qquad \boxtimes$$

We observe that for vectors as well, integration can be seen as the inverse of differentiation (cf. (12.8), where the one-dimensional analogue of the above is discussed).

A concrete example: A point moves in space with the velocity

$$\vec{v}(t) = \begin{pmatrix} 2t \\ 1 - 3t^2 \\ 1 + 4t^3 \end{pmatrix}.$$

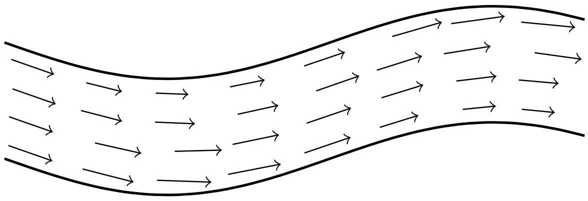
Where is the point at time $t = 2$, if at time $t = 0$ it was located a) at the origin, b) at the point $P(1, -1, 2)$?

(14.3) Vector fields (German: Vektorfelder)

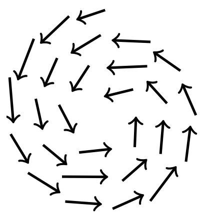
Think about the wind in $\mathbb{R}^3$, the flow of water in a river, blood in the bloodstream.

Until now we have considered vector functions $\vec{r}(t)$ for which the vector $\vec{r}$ only depended on a single variable, the parameter t. Now we want to examine vectors which depend on location (which can be described through three variables, namely the three coordinates). This situation can be represented by attaching to every point R in space the vector which corresponds to it. The lengths and directions of these vectors can vary from point to point. The examples below are two-dimensional simply for ease of illustration; in reality we should imagine these in a three-dimensional setting.

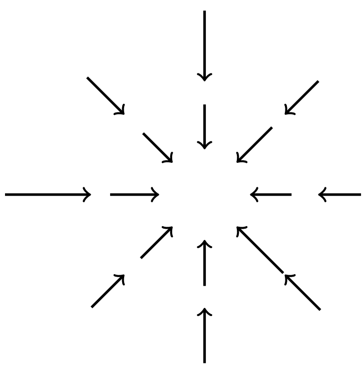
a) A liquid flows through a pipe. At every point R we can draw in a vector(!) representing the velocity of the flow at this location:



b) Wind velocity. For every point in a certain region of the atmosphere we draw in the corresponding wind velocity (here there seems to be a whirlwind in progress):



c) Force fields. This is an important application in physics. At every point in space (or in a region of space) a particular force acts, whose magnitude and direction in general depend on the point of action:



Now we consider the situation generally. In order to be able to express the point R with a vector, we choose an origin O. For every point R then, we have a location vector, namely the vector $\vec{r} = \overrightarrow{OR}$. To the point R we associate a vector $\vec{F}$ (for example, the direction and strength of the wind) which depends on R and thus on $\vec{r}$ (location), and so is written as $\vec{F}(\vec{r})$ (the wind at the location $\vec{r}$).

The result is a function which takes a vector $\vec{r}$ in space and associates with it a new vector

$$\vec{F}(\vec{r}),$$

i.e. a function which is defined on $\mathbb{R}^3$ and takes on values in $\mathbb{R}^3$:

$$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3.$$

Such a function is called a *vector field*. If the vector $\vec{F}$ represents a force, as in Example c), then it is also referred to as a *force field* (German: Kraftfeld).

The vector $\vec{F} = \vec{F}(\vec{r})$ is given by its three coordinate functions as usual:

$$\vec{F}(\vec{r}) = \begin{pmatrix} F_1(\vec{r}) \\ F_2(\vec{r}) \\ F_3(\vec{r}) \end{pmatrix}.$$

Here the functions' values $F_1(\vec{r})$, $F_2(\vec{r})$, $F_3(\vec{r})$ are real numbers which depend on the vector

$$\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

i.e. each of them depends on three real numbers. F_1, F_2, and F_3 are thus (real-valued) functions of three variables. Later we will discuss functions of multiple variables in more detail (Chapter 22). Two examples follow of vector fields which can be formulated algebraically:

1. If the functions involved are not so thoroughly understood, a table of values can be useful: here we consider a vector field in the plane, given by

$$\vec{F}(\vec{r}) = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} \qquad \text{for} \qquad \vec{r} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We produce a table of values, just as we would for an ordinary function.

$\vec{r}$	$\vec{F}(\vec{r})$	$\vec{r}$	$\vec{F}(\vec{r})$	$\vec{r}$	$\vec{F}(\vec{r})$
$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$
$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

2. Electrostatic field of a point charge: You do *not* need to know the following details from physics. We will simply take a closer look at the final formula, conduct a high-altitude analysis and in the next pages draw a few general conclusions from it.

How to start to understand a physics formula: investigate the dimensions, units, natural constants (incl. orders of magnitude). Let individual variables go to extremes $(0, \infty)$ and look what happens in the limit. Does 2π or 4π show up, if so why? Numeric examples, including some "strange" cases (for example if particular quantities/expressions are 0 - what does that mean?)

This is a general corollary to the last example. We observed that the magnitude of the electrostatic field is inversely proportional to the square of the magnitude of $\vec{r}$ (i.e. the distance from the origin):

$$|\vec{E}(\vec{r})| = \frac{Q}{4\pi\varepsilon_0|\vec{r}|^3}\ |\vec{r}| = \frac{Q}{4\pi\varepsilon_0|\vec{r}|^2}\ .$$

What we have elaborated here is a general principle which helps to explain many phenomena in the natural sciences and technology. Whenever we encounter an equation which says that some effect w satisfies such a law as this:

$$w = \frac{KQ}{4\pi r^2}\ ,$$

we can interpret it as follows: K is a natural constant, Q a source at the origin, r the distance from the origin and thus from the source. Does $4\pi r^2$ look familiar to us? That's the surface area of a sphere with radius r. But why has it turned up here in the denominator? If we place a radiating source, for example, at the origin (a mobile phone, a radioactive sample), we regard this as a point source which emits radially in all directions. Now if we only consider the attenuation of the radiation source due to distance, then the following is true: if you remove yourself from a distance of 2 metres to a distance of 4 metres from such a source (which, again, emits radially in all directions) then you will not receive half as much radiation from it but only a quarter as much: think of a sphere with radius 2 metres and one with radius 4 metres, both centred on the source. The surface areas of the spheres are $4\pi 2^2 = 16\pi$, respectively $4\pi 4^2 = 64\pi$: a factor of 4. The radiation originally emitted by the source is thus distributed over a 4 times larger surface, and correspondingly every cm^2 on the outer sphere receives only 1/4 the radiation that is seen by a cm^2 on the inner sphere. We can also understand the reason that $4\pi r^2$ appears in the denominator above: the quantity KQ is being distributed over a surface area of $4\pi r^2$, i.e. “per” unit area, and the “per” is an indication that something belongs in the denominator.

What are the consequences of such a formulation? A few examples follow, but first we repeat the limitations of the following conclusions: it has to be a point source, one which emits radially in all directions. We have not considered any braking effects, any interactions, but only the attenuation due to distance, purely geometrically and on the basis of symmetry. Physically this can often be explained by reference to the conservation of energy. Here we consider only the primary effect. For details and more precise information one should consult the corresponding subject literature.

1. Radiation from mobile phones: you should conduct most longer conversations over (wired) headphones: instead of holding the phone directly to your ear (and thus absorbing just about half its radiation with your head), at a distance of 1 metre your head and the sensitive brain region will receive a mere fraction of the radiation; the reduction is quadratic with increasing distance.
2. Similarly, for radioactivity *emitted by point sources*, a quadratic reduction occurs as in the first example. The 3 basic principles of radiation protection are then: shielding, minimising the duration of exposure, and maximising the distance.
3. Newton's law of gravitation:
$$F = \frac{Gm_1m_2}{r^2}.$$
4. The volume of a sound outdoors also decreases quadratically with the distance (if the sound comes from a point source - which of course loudspeakers typically are not!).

5. In the case of radar, one has to consider that the the waves reach the airplane and then are sent *again* (reflected) to reach the source. In actuality then (and without considering any special effects) this leads to an attenuation which follows

$$\frac{1}{r^2}\frac{1}{r^2} = \frac{1}{r^4}.$$

To overcome such extreme attenuation, the airplane can be fitted with a transponder of its own which sends an active signal.

6. In the field of seismology, subterranean explosions are used to obtain information about the subsurface. If the latter were homogeneous, when we set off a single explosive charge we would expect to observe attenuation that varies quadratically with distance to the first reflection; any deviations from that are a source of information to help seismologists draw conclusions about the subsurface.
7. Although the effect of a mid-air explosion also decreases quadratically, experience shows that the actual proportionality is approximately $r^{-2.2}$ thanks to the ground effect.
8. Explosions inside airplanes are particularly damaging: this can be explained by the fact an airplane is essentially a one-dimensional tube, and so the effect of the explosion can barely attenuate at all with distance.

A related question is how the effect of a stronger source can be neutralised by increasing the distance. But this can simply be read from the formula:

$$w = \frac{KQ}{4\pi r^2}\ .$$

K and 4π are constant. If we multiply Q by 9, we have to multiply r by 3 to get the same effect as before.

(14.4) Path integrals (German: Kurvenintegrale)

Preliminaries:

Implementation:

We can now summarise: the “work done” in the preceding situation is defined as

$$W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \dot{\vec{r}}(t)\, dt\ .$$

An integral of this form is called a path integral (sometimes also: line integral, curve integral, curvilinear integral). It can also be defined for arbitrary vector fields, independent of the concept of physical work. We record it then generally as follows:

Let $\vec{F} = \vec{F}(\vec{r})$ be any vector field and let C be a curve given by the parametric representation $\vec{r} = \vec{r}(t)$ ($t \in [a, b]$). By the *path integral* (or line integral, etc) of $\vec{F}$ over C we mean the integral

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \dot{\vec{r}}(t)\, dt\ . \tag{1}$$

Notice that (1) is in fact a completely ordinary integral of a function of one variable.

Now that we have defined a path integral, we can define the concept of work in its most general form as such a path integral — the motivating considerations above have shown that the path integral is indispensable for this purpose.

If we write the vectors in component form

$$\vec{F}(\vec{r}) = \begin{pmatrix} F_1(\vec{r}) \\ F_2(\vec{r}) \\ F_3(\vec{r}) \end{pmatrix}, \qquad \vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix},$$

then we obtain the following, written out in detail:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \dot{\vec{r}}(t)\, dt = \int_a^b \big(F_1(\vec{r}(t))\dot{x}_1(t) + F_2(\vec{r}(t))\dot{x}_2(t) + F_3(\vec{r}(t))\dot{x}_3(t)\big)\, dt\ . \tag{2}$$

We can use (2) to compute path integrals (see (14.6), which also presents a calculation scheme).

(14.5) Further information about path integrals

(14.6) Examples of the computation of path integrals

3. We wish to compute $\int_C \vec{r} \cdot d\vec{r}$, where C is the unit circle in the x-y–plane. For C we can take the parametrisation given in (8.2.2):

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} , \quad t \in [0, 2\pi] \; .$$

The vector field $\vec{F}(\vec{r})$ already appears here as the integrand, namely as $\vec{F}(\vec{r}) = \vec{r}$. In terms of the coordinates, then, simply

$$\vec{F}(\vec{r}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

So as a beginning, here we really need to find out what $\int_C \vec{r} \cdot d\vec{r}$ means:

Following equation (2), we have to compute

Our calculation scheme yields

$$\vec{F}(\vec{r}(t)) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} , \qquad \dot{\vec{r}}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} .$$

For the integrand $\vec{F}(\vec{r}(t)) \cdot \dot{\vec{r}}(t)$ we obtain

$$\cos t(-\sin t) + \sin t \cos t = 0 \; .$$

It follows that the path integral is also 0:

$$\int_C \vec{r} \cdot d\vec{r} = 0 \; . \qquad \boxtimes$$

The fact that the integrand is equal to zero has a very lucid explanation: Because the path C is a circle, the tangent vector $\dot{\vec{r}}$ is always perpendicular to the vector $\vec{r}$ $(= \vec{F}(\vec{r}))$. Their scalar product is therefore always zero. Even more lucidly, and with all due caution: if we interpret $\vec{F}$ as a force field, then during a circular motion, the force $\vec{F}(\vec{r}) = \vec{r}$ is always perpendicular to the "infinitely short" piece of path $d\vec{x}$. This means that the scalar product is $\vec{r} \cdot d\vec{r} = 0$ and therefore so is the physical work: $\int_C \vec{r} \cdot d\vec{r} = 0$.

A comment on the situation when the path is “closed”:

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.