13. FURTHER TECHNIQUES OF INTEGRATION

(13.2) Substitution

This method of integration is obtained through reversing the direction of the *chain rule* familiar from differential calculus. With somewhat different notation to that in (5.2) this rule becomes

$$F(u(x))' = F'(u(x)) \cdot u'(x) \ .$$

Now we substitute $F' = f$ into this formula. In particular, F becomes an antiderivative of f. Now the formula reads as follows:

$$F(u(x))' = f(u(x)) \cdot u'(x) \ .$$

We see that the composite function

$$x \mapsto F(u(x))$$

is an antiderivative of the (relatively complicated) function

$$x \mapsto f(u(x)) \cdot u'(x)$$

or in other words:

$$\boxed{\int f(u(x))u'(x)\,dx = F(u(x)) + C \ ,}$$

where F is an arbitrary antiderivative of f.

Despite the apparent complexity of its integrand, this formula has many important applications. To use it, we proceed as follows: Let $\int g(x)\,dx$ be the integral we want to compute. Now we look for a way to write the integrand $g(x)$ in the form

$$g(x) = f(u(x)) \cdot u'(x)$$

where the functions f and u are to be chosen appropriately. Above all we want to take care the function $u(x)$ and its derivative $u'(x)$ both appear in the integrand $g(x)$. A solid grasp of the rules for derivatives is indispensable here! Once we have determined f and u, then all we need to do is find an antiderivative F of f and apply it to $u(x)$. $F(u(x))$ is then an antiderivative of $g(x)$:

$$\int g(x)\,dx = \int f(u(x))\;u'(x)\,dx = F(u(x)) + C\;.$$

This procedure can be illustrated with a few examples.

(13.3) Examples of the substitution rule - in line with those in the textbook

1. $I = \displaystyle\int \cos(x^3)\cdot 3x^2\,dx\,.$

2. Und was macht man bei $J = \displaystyle\int \cos(x^3)\cdot x^2\,dx\,?$

3. For those who have serious difficulty in finding the right f, u, and u': write the integrand $g(x)$ explicitly as a product of 2 factors and then think about which of these factors can be an $f(u(x))$, if the other one has to be its $u'(x)$ - or vice versa. We will pay attention to this in the following exercises.
   We can even integrate such horrors as $\displaystyle\int \frac{\sin x}{\sqrt{\cos x}}\,dx$ in this way:

How do the physicists calculate here?

In practice we can summarise the procedure as follows:

1) Choose $u(x)$. Let $du = u'(x)\,dx$.
   (Formally this arises from the relationship

   $$\frac{du}{dx} = u'(x)$$

   by multiplying both sides by dx.)
2) In the desired integral, replace $u(x)$ with u and $u'(x)\,dx$ with du.
3) Now the integral has the form

   $$\int f(u)\,du\ .$$

   Find an antiderivative

   $$\int f(u)\,du = F(u) + C\ .$$

4) In the expression $F(u)$, replace u once again with $u(x)$. The desired integral is then equal to $F(u(x)) + C$.

We illustrate the process according to points 1) through 4) above with the following few examples.

4. $\displaystyle\int \sin x \cos x\,dx$

6. $\int \frac{1}{x^2} e^{1/x} dx$

(13.4) The substitution rule for definite integrals

Since $F(u(x))$ is an antiderivative of $f(u(x)) \cdot u'(x)$, it follows by the Fundamental Theorem of Calculus that

$$\int_a^b f(u(x))u'(x)\,dx = F(u(b)) - F(u(a))\ .$$

And since it is also the case that $F(u) = \int f(u)\,du$ (since F is an antiderivative of f), the right side can also be construed as

$$\int_{u(a)}^{u(b)} f(u)\,du \qquad \Big(= F(u(b)) - F(u(a))\Big)\ .$$

This gives us the formula

$$(*) \qquad \int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du\ .$$

Notice the "transformation" of the limits of integration. We have two possible approaches, then, which it is important not to confuse:

- If we regard x as the variable (left side of (*)), then at the end we need to migrate to $F(u(x))$. The limits of integration are then a and b.
- If we regard u as the variable betrachtet (right side of (*)), then at the end we stay with $F(u)$ but must replace the old limits of integration with the new ones, $u(a)$ resp. $u(b)$.

In this course we will work exclusively with the first variation. It's still a good idea to be aware of the second one, in order to be able to recognise it again in future courses. Critical for both forms is that we recognise the structure of the integrand. It needs to appear in this form: $g(x) = f(u(x)) \cdot u'(x)$ and we have to find the antiderivative F of f. The recommended procedure in this course for *definite* integrals with limits a and b is to separate the two steps clearly: find an antiderivative $F(u(x))$ first and then evaluate it at a and b using the formula

$$\int_a^b f(u(x))u'(x)\,dx = F(u(b)) - F(u(a))\ .$$

We revisit Example 6 above with some limits of integration:

$$\int_1^2 \frac{1}{x^2} e^{1/x} dx =$$

(13.5) Integration by parts (partielle Integration)

For this integration method we start with the *produkt rule* (5.2):

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \ .$$

Certainly every antiderivative of the left side is an antiderivative of the right side; equally, it's clear that $f(x)g(x)$ is an antiderivative of the left side. It follows that

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

or, written somewhat differently,

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx \ .$$

This is the formula for *integration by parts.*

As the letters f and g are often already in use for other things, we will typically write this formula with u and v:

$$\boxed{\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx \ .}$$

This formula can often be used successfully to find the integral of $\int f(x)\,dx$ when $f(x)$ can be written as a product of two factors, one of which (namely u') has an easy antiderivative (u), and the other of which (v) has an easy derivative (v'). We illustrate the process with a few examples.

Before we do that, we write down the formula for integration by parts as it applies to definite integrals. This is obtained simply by inserting the limits of integration:

$$\boxed{\int_a^b u'(x)v(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u(x)v'(x)\,dx \ .}$$

$\boxed{\text{(13.6) Examples of integration by parts}}$

1. $\displaystyle\int xe^x dx$

In the stochastics course MAT 183 we will need

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x} dx,$$

where $\lambda > 0$ (the expected value of an exponentially distributed random variable):

3. In (12.3) the antiderivative $\int \ln x\, dx$ seemed to appear from nowhere; how in the world could we have guessed that? Let's tackle it again now though:

LAPTE: When we integrate by parts, we need to proceed in such a way that the resulting new integral is easier than the old one. If we classify the functions involved according to LAPTE (Logarithms; Algebraic/Polynomials; Trigonometric; Exponential), then we need to take the derivative of whichever factor is further to the left in LAPTE, and integrate whichever factor is further to the right. For example it's useful to take the derivative of the natural logarithm, because then we obtain $1/x$, which we can combine with polynomials or other powers of x if we have any. Trigonometric and exponential functions, on the other hand, remain equally complicated if we have to integrate them.

(13.7) Table of integrals

Function	Antiderivative
a) Rational functions	
$(ax+b)^n \quad a \neq 0, \quad n \neq -1$	$\dfrac{(ax+b)^{n+1}}{a(n+1)}$
$(ax+b)^{-1} \quad a \neq 0$	$\dfrac{1}{a} \ln \lvert ax+b \rvert$
$\dfrac{1}{ax^2+2bx+c} \quad b^2 > ac$	$\dfrac{1}{2\sqrt{b^2-ac}} \ln \left\lvert \dfrac{ax+b-\sqrt{b^2-ac}}{ax+b+\sqrt{b^2-ac}} \right\rvert$
$\dfrac{1}{ax^2+2bx+c} \quad b^2 < ac$	$\dfrac{1}{\sqrt{ac-b^2}} \arctan \dfrac{ax+b}{\sqrt{ac-b^2}}$
$\dfrac{1}{ax^2+2bx+c} \quad b^2 = ac$	$-\dfrac{1}{ax+b}$
$\dfrac{ax+b}{cx+d} \quad c \neq 0$	$\dfrac{ax}{c} - \dfrac{ad-bc}{c^2} \ln \lvert cx+d \rvert$
b) Square roots	
$\sqrt{x^2 \pm a^2}$	$\dfrac{x}{2}\sqrt{x^2 \pm a^2} \pm \dfrac{a^2}{2} \ln \left\lvert x + \sqrt{x^2 \pm a^2} \right\rvert$
$\sqrt{a^2 - x^2}$	$\dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2} \arcsin \dfrac{x}{\lvert a \rvert}$
$\dfrac{1}{\sqrt{x^2 \pm a^2}}$	$\ln \left\lvert x + \sqrt{x^2 \pm a^2} \right\rvert$
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\arcsin \dfrac{x}{\lvert a \rvert}$
c) Trigonometric functions	
$\tan x$	$-\ln \lvert \cos x \rvert$
$\cot x$	$\ln \lvert \sin x \rvert$
$\dfrac{1}{\sin x}$	$\ln \lvert \tan \dfrac{x}{2} \rvert$
$\dfrac{1}{\cos x}$	$\ln \lvert \tan(\dfrac{x}{2} + \dfrac{\pi}{4}) \rvert$

Examples for the use of the table

A 13-4a)

A 13-5a)

So the methods for finding an antiderivative are: 1. substitution and integration by parts, 2. consult a table of integrals, 3. ask a mathematician, 4. numerical methods (cf. Chapter 21)

Differentiation always works:

when you think you've found an integral,

check by taking its derivative!

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.