 Chapter 20 will be discussed between Chapters 12 and 13; please bring the appropriate material to class.

12. ANTIDERIVATIVES AND THE INDEFINITE INTEGRAL

(12.1) Overview

With the *Fundamental Theorem of Calculus* the problem of computing definite integrals is reduced to that of finding antiderivatives. An antiderivative of $f(x)$ can also be written as *(11.4)*

$$\int f(x)\,dx\ ;$$

such an expression is called an *indefinite integral.* *(12.7)*

In this chapter the basic technical principles of integration are laid down and are illustrated by examples. *(12.4), (12.5)* *(12.6)*

(12.2) Recapitulation

Here we repeat the most important results from Chapter 11. As before, I denotes an arbitrary interval.

- The function F is called an antiderivative of $f : I \to \mathbb{R}$ if $F'(x) = f(x)$ for all $x \in I$.
- Two antiderivatives of $f : I \to \mathbb{R}$ differ only by an additive constant.
- The *Fundamental Theorem of Calculus*: If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)\ .$$

(12.3) Discussion of some antiderivatives

The simplest way of obtaining antiderivatives is clear: take a list of functions with their derivatives, and read it in the opposite direction. We illustrate this state of affairs using an extract from the table in (5.3):

Function $\longrightarrow$	Derivative
Antiderivative $\longleftarrow$	Function
x	1
x^2	$2x$
x^r	rx^{r-1} $(r \in \mathbb{R})$
e^x	e^x
$\ln x$	$1/x$
$\sin x$	$\cos x$

Some basic initial observations:

What happens with the natural logarithm function, downward?

What happens with the natural logarithm function, upward?

(12.4) A first list of antiderivatives

On the basis of the various observations we made in (12.3), we can improve the table above for practical use:

Function $f(x)$	An antiderivative $F(x)$
a	ax
$x^r \quad (r \neq -1)$	$\dfrac{x^{r+1}}{r+1}$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$

An important fact is that an antiderivative is only determined up to an additive constant. This is why, for the sake of clarity, we often write

$$F(x) + C, \quad \text{so e.g.} \quad \ln|x| + C \quad \text{etc.}$$

In the above table this has been omitted. In some applications this so-called *constant of integration* must absolutely not be forgotten, e.g. for differential equations (Chapter 16); in others it is not necessary, e.g. for the use of the Fundamental Theorem (11.4), since there F can be any arbitrary antiderivative:

(12.5) Integration rules

The table in (12.4) was constructed by reversing the formulas for derivatives in (5.3). The rules for derivatives in (5.2) can also be reinterpreted to yield rules for integration. As the derivative of a sum equals the sum of the derivatives, so the antiderivative of a sum is the sum of the antiderivatives. Individually, we arrive in this way at the following rules:

Let $F(x)$ resp. $G(x)$ be antiderivatives of $f(x)$ resp. of $g(x)$. Then:

(1) $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$.

(2) $F(x) - G(x)$ is an antiderivative of $f(x) - g(x)$.

(3) $cF(x)$ is an antiderivative of $cf(x)$.

If we apply these rules to the computation of integrals using the Fundamental Theorem of Calculus, we obtain the following formulations already mentioned in (10.8):

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx\ .$$

We mention as a *warning* that there is no corresponding formula for the product

$$\int_a^b f(x)g(x)\,dx\ .$$

Nor, analogously, is there one for the quotient:

(12.6) Examples, varied from the book

Find $\displaystyle\int_1^2 (3x^4 + 3x^2 - 2)\,dx$.

Compute $\int_1^2 \left(\sqrt{t} + \frac{2}{t^4}\right) dt$.

Find the area of the region below the graph of $y = \dfrac{1}{x}$ on the interval $[1, 2]$. What happens if we replace $[1, 2]$ with $[-2, -1]$ (algebraically and graphically)?

Example from physics: work of expansion of a gas

(12.8) Integration as the opposite of differentiation

If the integrand itself is already given as the derivative of a function, then the definite integral can be written down immediately with the help of the Fundamental Theorem since $f(x)$ is certainly an antiderivative of $f'(x)$:

$$\int_a^b f'(x)\,dx = f(b) - f(a)\ .$$

<u>Example</u>

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.