11. THE FUNDAMENTAL THEOREM OF CALCULUS

(11.2) The definite integral as a function of the upper limit

Here and in the following we will use I to denote an arbitrary (open, closed, etc.) interval. We consider a continuous function

$$f : I \to \mathbb{R}$$

and choose a fixed $a \in I$. For every $x \in I$, then, the definite integral

$$\int_a^x f(t)\,dt$$

is defined according to (10.2). In this way we obtain a new function

$$\Phi : I \to \mathbb{R}\ , \quad \Phi(x) := \int_a^x f(t)\,dt\ .$$

If $f(x) \geq 0$, then $\Phi(x)$ has a simple geometric meaning as the area of the shaded region:

This is the case for $x > a$ at least; in accordance with (10.8) we see that $\Phi(a) = 0$ ("vanishing" area), and for $x < a$, $\Phi(x)$ is the negative area.

Example: In (10.7) we saw that:

$$\int_0^b x^2\,dx = \frac{b^3}{3}\ .$$

After a change of notation we obtain in this particular case a formula for the function Φ, namely

$$\Phi(x) = \int_0^x t^2\,dt = \frac{x^3}{3}\ .$$

Reverting to generality: If the function $\Phi(x) = \int_a^x f(t)\,dt$ is defined in this way, then the following extremely important statement is true:

Fact (I): Φ is differentiable on I and $\Phi'(x) = f(x)$ for all $x \in I$.

The only example until now of an integral for whose value we have a formula is the one mentioned above. Here we have $f(x) = x^2$, $\Phi(x) = x^3/3$, and in fact $\Phi'(x) = f(x)$. This corroborates Fact (I) above.

Now we will give an intuitive justification for Fact (I):

(11.3) Antiderivatives

Building on Fact (I) from (11.2), we introduce a new and important concept:

Let $f : I \to \mathbb{R}$ be a function. By an *antiderivative* (also sometimes: primitive function, primitive integral, or indefinite integral) of f, we mean a function F such that:

$$F'(x) = f(x) \quad \text{for all} \quad x \in I \ .$$

Now Fact (I) can also be formulated as follows: if f is continuous, then the function Φ defined as

$$\Phi(x) = \int_a^x f(t)dt$$

is an antiderivative of f.

It follows that every continuous function f has (at least) one antiderivative, namely Φ. Since directly computing a definite integral is quite laborious, though, this is in the short term only of theoretical interest.

The pivotal insight here is that one can also look at the problem the other way round. In many cases, we can write down an antiderivative immediately, simply by reversing the rules for the derivative. So for example, $F(x) = \sin x$ is an antiderivative of $f(x) = \cos x$ (since $F'(x) = (\sin x)' = \cos x = f(x)$), $G(x) = x^2$ is an antiderivative of $g(x) = 2x$ (since $G'(x) = (x^2)' = 2x = g(x)$). Let's practice this approach right now:

We can use this state of affairs to compute definite integrals by relating them to antiderivatives which we have found in some other way. We apply these considerations to the first example. In general, we know that in theory

$$\Phi(x) = \int_a^x \cos t \, dt$$

is an antiderivative of $f(x) = \cos x$. On the other hand, we have just seen that

$$F(x) = \sin x$$

is also an antiderivative of $\cos x$ ist. If we could now conclude that these two antiderivatives Φ and F were the same, i.e. that

$$\int_a^x \cos t \, dt = \sin x \quad \text{or, specifically,} \quad \int_a^b \cos t \, dt = \sin b$$

then we would have an extremely easy method for computing an integral which was until now so inaccessible. Unfortunately this is not quite completely the case (take comfort though, all is not lost!).

The reason that the observation immediately above is not correct is that a continuous function always has infinitely many antiderivatives. In fact, the following is true:

Fact (II): If $F(x)$ is an antiderivative of $f(x)$, then for any real number C, $F(x)+C$ is also an antiderivative of $f(x)$.

This follows simply from the fact that the derivative of any constant function is 0, and so indeed $(F(x) + C)' = F'(x) = f(x)$.

The converse of this observation, however, is also true:

Fact (III): Let $F_1(x)$ and $F_2(x)$ be antiderivatives of $f(x)$. Then it holds that $F_1(x) - F_2(x) = C$ (for all x) for some suitable constant C, or to put it another way: $F_1(x) = F_2(x) + C$.

Proof:

Now we can return to contemplating our previous example with new courage. It's true that we cannot conclude that the two antiderivatives $\Phi(x)$ and $\sin x$ of $f(x) = \cos x$ are equal. What we can say though is that they differ from each other only by a constant. There is, then, a number C such that

$$\Phi(x) = \int_a^x \cos t \, dt = \sin x + C \ .$$

How can this C be determined? If we let $x = a$, then

$$\Phi(a) = \int_a^a \cos t \, dt = 0 = \sin a + C$$

and we find

$$C = -\sin a \ .$$

For all x, then, it holds that

$$\int_a^x \cos t \, dt = \sin x - \sin a$$

and in particular (for $x = b$)

$$\int_a^b \cos t \, dt = \sin b - \sin a \ .$$

In even greater particularity, we obtain for example the result

$$\int_0^{\pi/2} \cos t \, dt = \sin \frac{\pi}{2} - \sin 0 = 1 \ .$$

Now this is a real success! Here is a definite integral, so complicated in its definition (with Riemann sums), and yet we have been able to compute its value in a very simple way.

Interpreting the last formula geometrically, we see that the area of the shaded region under the cosine curve is $= 1$!

Let's try the same approach again: we want to calculate

$$\int_a^b t^3 \, dt \ .$$

(11.4) The fundamental theorem of calculus

The following result is extremely important for computing integrals.

The Fundamental Theorem of Calculus:

Let f be a function, defined and continuous on the interval I, and let F be an arbitrary antiderivative of f. Then for all $a, b \in I$:

$$\int_a^b f(x)\,dx = F(b) - F(a)\ .$$

The proof of this theorem follows the line of reasoning we have already employed. According to "Fact (I)" from (11.2), $\Phi(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$. According to "Fact (III)" from (11.3), then, Φ and F only differ by a constant C:

$$\int_a^x f(t)\,dt = F(x) + C \quad \text{for all} \quad x \in I\ .$$

Since $\Phi(a) = 0$, we have $F(a) + C = 0$, i.e. $C = -F(a)$. For $x = b$ we obtain

$$\int_a^b f(t)\,dt = F(b) - F(a)\ .$$

Up to the (insignificant) difference in choice of integration variable, this is precisely the claim of the theorem. We emphasise again that the formulation of the Fundamental Theorem is valid for $a > b$ as well as for $a < b$ (and trivially for $a = b$).

Notice that the Fundamental Theorem creates a close relationship between the derivative and the integral, a relationship which, based strictly on their respective definitions, is not at all obvious (although English does rather give the game away by referring to the integral as the *anti*derivative!).

For the frequently occurring difference $F(b) - F(a)$ one often uses the abbreviation

$$F(b) - F(a) = F(x)\Big|_a^b \text{ or } \Big[F(x)\Big]_a^b\ .$$

For example:

$$x^2\Big|_a^b = b^2 - a^2\ .$$

In the next chapter we will make systematic use of the Fundamental Theorem. We close this chapter with one more simple example, which also explains a further bit of notation. We consider the constant function $f(x) = 1$. Rather than $\int_a^b 1\,dx$ one simply writes $\int_a^b dx$. Since it is clear that $F(x) = x$ is an antiderivative of $f(x)$, we obtain

$$\int_a^b dx = x\Big|_a^b = b - a\ .$$

Geometrically this is the area of the rectangle with height 1 over the interval bounded by a and b (if $a < b$; otherwise $b - a < 0$, and we get the negative area).

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.