10. THE DEFINITE INTEGRAL

(10.2) Definition of the definite integral

As alluded to in (9.6), our plan here is to introduce a construction which generalises the process we investigated in the examples of Chapter 9.

To that end, consider a closed interval $[a, b]$ and a *continuous function*

$$f : [a, b] \to \mathbb{R}, \quad (a < b) \ .$$

We proceed in a series of steps:

Step 1

We divide the interval $[a, b]$ into n *subintervals* by choosing a set of points

$$a = x_0 < x_1 < x_2 < \ldots < x_{i-1} < x_i < \ldots < x_n = b \, .$$

(We may also speak of a *subdivision* of the interval $[a, b]$.) The i–th subinterval $[x_{i-1}, x_i]$ has length $\Delta x_i = x_i - x_{i-1}$. (It is not necessary that the subintervals all have the same length.)

Step 2

In each subinterval $[x_{i-1}, x_i]$ we choose an *intermediate point* ξ_i:

$$x_{i-1} \le \xi_i \le x_i \ .$$

Step 3

Using the values just chosen, we form the sum

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \ .$$

Such a sum is called a *Riemann sum* (after B. Riemann, 1826–1866).

Step 4

Now we make the subintervals smaller and smaller (i.e. we let $\Delta x_i \to 0$). As this happens, the number n of subintervals automatically increases ($n \to \infty$).

Through this process the Riemann sum approaches a limit. This limit is called the *definite integral of f from a to b* and we use

$$\boxed{\int_a^b f(x)\,dx}$$

to denote it. Formally written down:

$$\boxed{\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \ .}$$

In this way the definite integral is introduced as a general mathematical construction. A comparison with (9.2) through (9.5) will show that the observations we made there are concrete applications of the abstract concept of the integral (see also (10.5)).

* Important observations in the textbook (read at home): (10.3), (10.5), and (10.6a)

* for instance, answers to "what exactly happens 'at infinity'?": no esoterica required, just study mathematics to learn the whole story.

* $\int_a^b f(x)\,dx$ is a number, if a and b are given. An integral is called a *definite* integral if the limits of integration are given.

* The variable of integration can be arbitrarily changed and disappears in the calculation; the following expressions are all equal, no matter whether we work with x, u, or s:

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(s)\,ds.$$

* Please NEVER write

$$\int_a^x f(x)\,dx$$

The *limit* of integration should NEVER be the *variable* of integration!!!!

(10.6) Further examples of applications of integrals

This section complements the examples in Chapter 9. We want to demonstrate again that the concept of an integral is essential for the description of certain facts.

b) Length of a curved segment

Length of a curved segment, continued:

(10.7) An example of the computation of a definite integral

Until now we have not explicitly computed a single integral; we will do that *systematically* later. Right now let's just compute the integral of x^2 "by hand". A related article of mine (in German) about a famous formula by Gauss: `https://schweizermonat.ch/der-kleine-gauss-laesst-einem-keine-ruhe` .

Integral of x^2 "by hand", continued:

(10.8) A few rules for calculating the definite integral

a) Sums, differences, multiples

The following rules hold for the definite integral:

$$(1)\quad \int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$(2)\quad \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx\ .$$

b) Exchanging the limits of integration

The integral $\int_a^b f(x)\,dx$ was defined under the explicit condition that $a < b$. For $a \geq b$ we *define* the following, largely in accord with intuition:

$$(3)\quad \int_a^a f(x)\,dx = 0$$

$$(4)\quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad \text{für} \quad b < a\ .$$

(Exchanging the limits of integration results in a sign change!)

c) Splitting the interval

Let $c \in [a, b]$. Then it holds that

$$(5) \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \ .$$

If one considers the special case in which an integral represents area, this result becomes clear (although it can also be proven purely computationally):

Additionally, the above formula holds for a, b, c in arbitrary order, i.e. for $c < b < a$ etc. This can be proven by consideration of the various possible cases.

Important:

1. Next, read the entire corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.