

This script is an extract, with gaps, from the book “Introduction to the Mathematical Treatment of the Natural Sciences I - Analysis” by Christoph Luchsinger and Hans Heiner Storrer, Birkhäuser Scripts. As a student you should buy the book as well and work your way through it completely during the course MAT 182. You are allowed to save this PDF and modify it as you like, for your own use during MAT 182. For further use outside of MAT 182, please contact the lecturer, Christoph Luchsinger of the University of Zürich, in advance. The copyright remains with Birkhäuser!

For ease of orientation, the same numbering scheme is maintained here as in the book.

A few philosophical considerations:

Lemma      Proposition      Theorem      Fundamental theorem      -      Corollary

In the lecture, the exercises, and the exam, numbers will be rounded to 5 significant digits: 2.42567 is rounded to 2.4257

**LS:** Luchsinger-Storrer; p 8, l-5: page 8, 5th row from the bottom; p +3: 3 pages further

# A. VECTOR ARITHMETIC

## 1. VECTORS AND THEIR GEOMETRIC MEANING

(1.2) Introductory examples
-----------------------------

Quantities which can be completely described by giving a number (along with a corresponding unit of measurement):

- price of an item for sale (in francs),
- length of a line segment (in cm),
- body temperature (in °C),
- frequency of a radio broadcast (in MHz),
- population of a city (a pure number, with no unit of measurement).

There are other quantities which are not completely determined by giving a single number. Among these are properties which also depend on the direction.

- velocity of a vehicle
- force acting on a point mass
- a point's location in space relative to a fixed initial point
- a translation in geometry

**Such a directed line segment is called a vector**

There are evidently two variations on the concept of a vector: location vectors (force - it matters where, i.e. at what location, the force acts, where the initial point is) and free vectors (the initial point is irrelevant). Because its initial point is irrelevant, a free vector is not uniquely determined. Diagrams and a definition:

## (1.6) Addition and subtraction of vectors

a) Motivation

Experiments show that forces and velocities “add” according to a so-called “parallelogram rule”.

(i) Parallelogram of forces:

(ii) If an airplane has velocity  $\vec{v}$  relative to the air (true airspeed TAS), and the air has velocity  $\vec{w}$  relative to the ground, then the velocity  $\vec{r}$  of the airplane relative to the ground can be read from the following diagram:

(iii) Something analogous holds in geometry for translations: a translation by  $\vec{u}$ , followed by a translation by  $\vec{v}$  is equivalent to a translation by  $\vec{w}$ , according to the following sketch:

b) Definition of addition

In keeping with the examples above (at first (ii) and (iii), (i) will come into its own afterward) we define the *addition* of vectors as follows:

Let  $\vec{a}$  and  $\vec{b}$  be two (free) vectors. We choose an arbitrary representative of  $\vec{a}$  and attach to its terminal point  $P$  the particular representative of  $\vec{b}$  whose initial point is at  $P$ .

We define the *sum vector*  $\vec{a} + \vec{b}$  of  $\vec{a}$  and  $\vec{b}$  as the vector which in this situation leads from the initial point of  $\vec{a}$  to the terminal point of  $\vec{b}$ , resp. (as we are dealing with free vectors) every vector which is parallel to it (and has the same orientation).

In the construction above  $\vec{a}$  and  $\vec{a} + \vec{b}$  have the same initial point while  $\vec{b}$  is translated. The following variation, which is constructed in imitation of the parallelogram of forces and which naturally yields the same result, employs only location vectors with the initial point  $O$ .

Let  $\vec{a} = \overrightarrow{OP}$ ,  $\vec{b} = \overrightarrow{OQ}$  be two vectors. Then

$$\vec{a} + \vec{b} = \overrightarrow{OR},$$

where  $R$  is the fourth vertex of the parallelogram whose other vertices are  $O$ ,  $P$ ,  $Q$ :

c) The opposite vector

In general, for a vector  $\vec{a}$  we define the *opposite vector*  $-\vec{a}$  as the vector which has the same length as  $\vec{a}$  but has the opposite direction:

$$\begin{aligned}\vec{a} &= \overrightarrow{PQ} \\ -\vec{a} &= \overrightarrow{QP}\end{aligned}$$

It is immediately clear then from the definition of vector addition that

$$\vec{a} + (-\vec{a}) = \vec{0} \quad \text{und} \quad (-\vec{a}) + \vec{a} = \vec{0}.$$

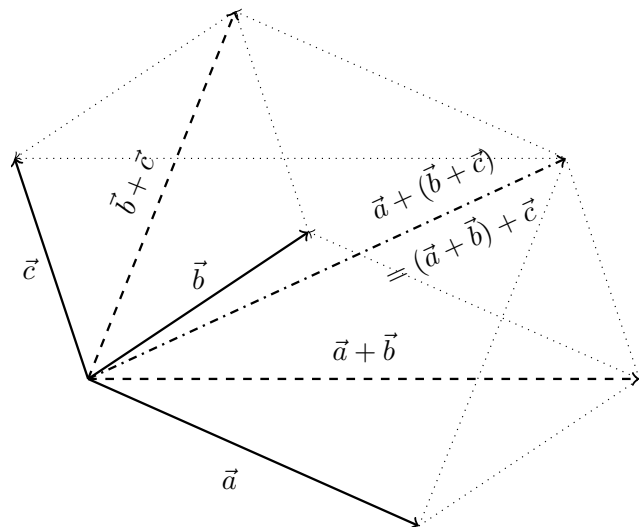
d) Rules for the addition of vectors

The addition of vectors, as we have defined it above, satisfies exactly the same basic rules as the addition of numbers, namely:

- (1) Commutative property:  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ , for all vectors  $\vec{a}, \vec{b}$ .
- (2) Associative property:  $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ , for all vectors  $\vec{a}, \vec{b}, \vec{c}$ .
- (3) There exists a *neutral element* for the operation of addition, namely the zero vector  $\vec{0}$ , which has the following property:  
 $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$ , for all vectors  $\vec{a}$ .
- (4) For every vector  $\vec{a}$  there exists a unique *opposite vector*  $-\vec{a}$  with the following property:  
 $\vec{a} + (-\vec{a}) = \vec{0}$  (and analogously  $(-\vec{a}) + \vec{a} = \vec{0}$ ).

A warning, with illustrations

It should be mentioned here that in general the magnitude of the sum of vectors is different from the sum of the magnitudes:  $|\vec{a} + \vec{b}| \neq |\vec{a}| + |\vec{b}|$ . As can be seen immediately from the geometric definition, the so-called *triangle inequality* does however hold:  $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$ , see also (26.8).



e) Subtraction of vectors

The *difference* of two vectors is defined using already-established concepts. In analogy to the subtraction of numbers, we set

$$\vec{a} - \vec{b} := \vec{a} + (-\vec{b}).$$

The geometric interpretation is as follows: we move  $\vec{a} - \vec{b}$  to the terminal point of  $\vec{b}$  and obtain the following diagram:

It is worth restating this fact in a sentence:

$\vec{a} - \vec{b}$  is the vector which goes from the terminal point of  $\vec{b}$  to the terminal point of  $\vec{a}$  (pay attention to the order!), provided that  $\vec{a}$  and  $\vec{b}$  have the same initial point.

(1.7) Multiplication of a vector with a scalar

Let  $r \in \mathbb{R}$  (these notations are explained in (26.2) and (26.3)) be a scalar (real numbers which come up in connection with vectors are often called *scalars* to distinguish them), and let  $\vec{a}$  be a vector. We define the vector  $r\vec{a}$  by distinguishing three cases:

- $r > 0$  : Here  $r\vec{a}$  is the vector which has the same direction as  $\vec{a}$  and whose magnitude is  $r$  times the magnitude of  $\vec{a}$ :

$$|r\vec{a}| = r|\vec{a}| .$$

- $r = 0$  :  $r\vec{a} = 0\vec{a}$  is defined to be the zero vector:

$$0\vec{a} := \vec{0} .$$

- $r < 0$  : Here  $r\vec{a}$  is the vector which has the opposite direction to  $\vec{a}$  and whose magnitude is  $|r|$  times the magnitude of  $\vec{a}$ :

$$|r\vec{a}| = |r||\vec{a}| = (-r)|\vec{a}| .$$

The magnitude  $|r|$  is explained in (26.8).

The vector  $r\vec{a}$  is also referred to as a *multiple* of  $\vec{a}$ . The multiplication we have just described, of a vector with a number, satisfies the following rules, which can be proven geometrically without too much effort.

$$r(\vec{a} + \vec{b}) = r\vec{a} + r\vec{b}$$

$$(r + s)\vec{a} = r\vec{a} + s\vec{a}$$

$$r(s\vec{a}) = (rs)\vec{a}$$

$$1\vec{a} = \vec{a}$$

$$(-1)\vec{a} = -\vec{a}$$

for all vectors  $\vec{a}, \vec{b}$  and all scalars  $r, s$ .

A vector is called a *unit vector* if its magnitude is 1. Any vector  $\vec{a} \neq \vec{0}$  can be made into a unit vector  $\vec{a}_1$  by dividing it by  $|\vec{a}|$  (i.e. by multiplying it with the reciprocal of



the number  $|\vec{a}|$ ):

$$\vec{a}_1 := \frac{1}{|\vec{a}|} \vec{a} \quad \text{is a unit vector, since} \quad |\vec{a}_1| = \left| \frac{1}{|\vec{a}|} \vec{a} \right| = \frac{1}{|\vec{a}|} |\vec{a}| = 1 .$$

(1.8) The scalar product (also: dot product, inner product; German: Skalarprodukt)

a) Motivation from physics

A point mass moves in a straight line and is acted on by a constant force applied parallel to the direction of motion.

The work done by the force in this case is defined by the formula

$$\text{Work} := \text{force times distance traveled.}$$

Here the formula  $W = Fs$  is familiar from secondary school physics, although this would perhaps be more properly stated with the use of vectors. This formula is no longer valid though in the more general case, when the force does not act parallel to the direction of motion. In order to capture this correctly algebraically, we first consider the following right triangle. It is clear that  $\cos \varphi = \frac{u}{F}$ , thus  $u = F \cos \varphi$ .

With this we are ready to derive the more general formula:

$$\vec{F} = \text{force.}$$

$$\vec{s} = \text{directed distance} \\ \text{which the point mass} \\ \text{has traveled.}$$

Here the force does not act with its full magnitude  $|\vec{F}|$ , but rather with the magnitude of its projection onto the direction of the vector  $\vec{s}$ . As this projection has the magnitude  $|\vec{F}| \cos \varphi$ , the work  $W$  done by the force is:

$$W = \text{distance times "projected force"} = |\vec{s}| |\vec{F}| \cos \varphi .$$

It should also be pointed out that  $\cos \varphi$  is negative for  $\frac{\pi}{2} < \varphi < \frac{3\pi}{2}$ . In consequence of this,  $W$  is either positive or negative, according to whether the projection of  $\vec{F}$  points in the direction of  $\vec{s}$  or in the opposite direction.

Review: Radians / trigonometric functions

b) Definition of the scalar product

Proceeding from the above motivation, we define the *scalar product* of any two vectors  $\vec{a}$  and  $\vec{b}$  by the formula

$$\vec{a}\vec{b} := |\vec{a}| |\vec{b}| \cos \varphi ,$$

where  $\varphi$  is the angle between  $\vec{a}$  and  $\vec{b}$ .

Sometimes this is also written as  $\vec{a} \cdot \vec{b}$  instead of  $\vec{a}\vec{b}$  (hence its alternative name of “dot product”).

c) Observations

- 1) It should be emphasised that the scalar product of two vectors is a *number* (i.e. a scalar).
- 2) If  $\vec{a}$  or  $\vec{b}$  is the zero vector, then the angle  $\varphi$  is undefined. In this case we reasonably set  $\vec{a}\vec{b} = 0$ .
- 3) The angle  $\varphi$  between two vectors is not uniquely determined:

Fortunately, however,  $\cos \varphi = \cos(2\pi - \varphi)$  and so the value of the scalar product does not depend on which angle is chosen.

- 4) For  $0 \leq \varphi \leq 2\pi$ , the value of  $\cos \varphi$  is zero if and only if  $\varphi = \frac{\pi}{2}$  or  $\varphi = \frac{3\pi}{2}$ . Therefore the following holds for  $\vec{a} \neq \vec{0}$  and  $\vec{b} \neq \vec{0}$ :

The scalar product  $\vec{a}\vec{b}$  is equal to zero if and only if  $\vec{a}$  and  $\vec{b}$  are perpendicular to each other.

- 5) Since  $\cos 0 = 1$ , the scalar product of a vector with itself is given by

$$\vec{a}\vec{a} = |\vec{a}|^2 \quad \text{or in shorter notation}$$

$$\vec{a}^2 = |\vec{a}|^2 .$$

- 6) Here it should be particularly noted that the scalar product cannot be “canceled” or “divided”: if  $\vec{a}\vec{b} = \vec{a}\vec{c}$ , in general it does not follow that  $\vec{b} = \vec{c}$ . (For example, there are many different vectors  $\vec{v}$  which are perpendicular to  $\vec{a}$ ; for all of these we have  $\vec{a}\vec{v} = 0$ .)

d) Rules for the scalar product

The following rules hold for the scalar product:

- (1)  $\vec{a}\vec{b} = \vec{b}\vec{a}$ , for all  $\vec{a}, \vec{b}$  (commutative property).  
 (2)  $(r\vec{a})\vec{b} = r(\vec{a}\vec{b})$ , for all  $\vec{a}, \vec{b}$  and all  $r \in \mathbb{R}$ .  
 (3a)  $(\vec{a} + \vec{b})\vec{c} = \vec{a}\vec{c} + \vec{b}\vec{c}$ ,  
 (3b)  $\vec{a}(\vec{b} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}$ , for all  $\vec{a}, \vec{b}, \vec{c}$  (distributive properties).

These rules can be justified as follows:

- (1) This follows immediately from the definition.  
 (2) Here we need to consider two cases.

For  $r \geq 0$  we have  $(r\vec{a})\vec{b} = |r\vec{a}| |\vec{b}| \cos \varphi = r|\vec{a}| |\vec{b}| \cos \varphi = r(\vec{a}\vec{b})$ .

For  $r < 0$  the vector  $r\vec{a}$  has the opposite direction to  $\vec{a}$ . The angles are related as shown in the following figure:

- (3) A justification for the distributive properties is more complicated. You will find it in the appendix (27.2).

(1.9) The vector product (also: cross product; German: Vektorprodukt, Kreuzprodukt)

a) Definition of the vector product

Along with the product we have just discussed — the scalar product of two vectors, which is a real number — there exists another way to combine two vectors together, namely the *vector product*, which associates to any two vectors  $\vec{a}$  and  $\vec{b}$  a new vector  $\vec{a} \times \vec{b}$ . The definition which follows may seem a little arbitrary at first glance, but it will reveal itself to have applications in many settings.

Here we state the definition of the vector product.

Let  $\vec{a}$  and  $\vec{b}$  be vectors, and let the angle between them  $\varphi$  be defined in such a way that  $0 \leq \varphi \leq \pi$ . The vector product  $\vec{a} \times \vec{b}$  is defined as the vector which has the following three properties:

- (A) Direction:  $\vec{a} \times \vec{b}$  is perpendicular (normal, orthogonal) to the plane defined by  $\vec{a}$  and  $\vec{b}$ .
- (B) Orientation: The vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{a} \times \vec{b}$ , in this order, form a *right-hand system*. By this we mean: imagine that you hold your right hand out flat, palm up, thumb and index finger extended, and raise the middle finger straight up so that it is perpendicular to them both. If the thumb points in the direction of  $\vec{a}$  and the index finger in the direction of  $\vec{b}$ , then the middle finger points in the direction of  $\vec{a} \times \vec{b}$ .
- (C) Magnitude: The magnitude of  $\vec{a} \times \vec{b}$  satisfies

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \varphi .$$

1. The significance of the vector product lies primarily in the fact it enables us to systematically produce a vector perpendicular to two given vectors.
2. This is very practical for describing certain phenomena in physics: angular momentum, torque, and the Lorentz force, to name a few.
3.  $|\vec{a} \times \vec{b}|$  is equal to the area of the parallelogram spanned by  $\vec{a}$  and  $\vec{b}$ , which has a side of length  $|\vec{a}|$  and an altitude of  $|\vec{b}| \sin \varphi$ .

c) Rules for the vector product

The following rules hold for the vector product:

- (1)  $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$ , for all vectors  $\vec{a}, \vec{b}$ .
- (2)  $(r\vec{a}) \times \vec{b} = \vec{a} \times (r\vec{b}) = r(\vec{a} \times \vec{b})$ , for all  $\vec{a}, \vec{b}$  and all  $r \in \mathbb{R}$ .
- (3)  $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ ,  
 $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ , for all vectors  $\vec{a}, \vec{b}, \vec{c}$ .

Unexpected here is the fact that due to (1), the commutative property does *not* hold (one refers to this as “anticommutativity”). This will oblige us to use a certain caution in dealing with the vector product. (3) are the distributive properties. It should be emphasised here that with the vector product — exactly as with the scalar product — “cancellation” is not possible.

## (1.10) Applications in geometry

a) Parametric representation of a line

Let two points  $A, B$  in space ( $A \neq B$ ) be given. The goal is to represent vectorially the line  $g$  through  $A$  and  $B$ . This is done in essence by presenting the condition which a point  $R$  must satisfy if it lies on the line.

To begin with, we choose some point  $O$  as the origin and consider the location vectors

$$\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{OB}, \vec{r} = \overrightarrow{OR}.$$

Then  $\vec{b} - \vec{a} = \overrightarrow{AB}$ , and this (free) vector gives us the direction of the desired line  $g$ .

If we extend from  $A$  an arbitrary multiple  $t(\vec{b} - \vec{a})$ ,  $t \in \mathbb{R}$ , then we obtain a point on the line  $g$ ; conversely, every point on  $g$  can be obtained in this way.

So we have shown that the point  $R$  lies on  $g$  if and only if the location vector  $\vec{r} = \overrightarrow{OR}$  satisfies:

(\*)

$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a}),$$

where  $t$  is a real number.

The quantity  $t$  is called a *parameter* and the equation (\*) is said to be a *parametric representation* of  $g$ . The indefinite article here is justified by the fact that a line  $g$  has infinitely many different parametric representations: if we simply choose, not  $A$  and  $B$ , but any two other points on  $g$ , then the quantities  $\vec{a}$  and  $\vec{b}$  in (\*) will be different.

If we interpret  $t$  as the time, we arrive at a “dynamic” interpretation of (\*). As  $\vec{r}$  depends on  $t$ , it may also be written as  $\vec{r}(t)$ :

$$\vec{r}(t) = \vec{a} + t(\vec{b} - \vec{a}) .$$

This formulation can be used to describe a linear motion in space: at time  $t$  the moving object is located at the terminal point of the location vector  $\vec{r}(t)$ . In particular we have  $\vec{r}(0) = \vec{a}$ ,  $\vec{r}(1) = \vec{b}$ , i.e. at time  $t = 0$  the object is at point  $A$ , at time  $t = 1$  at point  $B$ . In (8.3) we will see how parametric representations can in fact be used to describe motion along arbitrary curves.

Finally, it should be mentioned that the line  $g$  can also be described by specifying a point  $A$  and a direction vector  $\vec{q}$  ( $\vec{q} \neq \vec{0}$ ). In this setting one simply replaces  $\vec{b} - \vec{a}$  with  $\vec{q}$  in the equation (\*):

$$\vec{r} = \vec{a} + t\vec{q} .$$

b) The plane normal to a vector, passing through a point

Consider a given point  $A$  and a direction, represented by the (free) vector  $\vec{n}$ . We desire to find the plane normal to  $\vec{n}$  which passes through  $A$ .

To solve this exercise we choose once again an origin  $O$ . As you can observe from the following sketch, the terminal point  $R$  of the vector  $\vec{r} = \overrightarrow{OR}$  lies in the desired plane if and only if  $\vec{r} - \vec{a}$  (with  $\vec{a} = \overrightarrow{OA}$ ) is perpendicular to  $\vec{n}$ . According to Observation 4) in (1.8.c) though, this is the case if and only if:

$$\vec{n}(\vec{r} - \vec{a}) = 0 .$$

The points  $R$  of this plane are thus described.



c) The plane passing through three points

Three points  $A, B, C$  which are not collinear determine a plane; we would like to describe this plane vectorially. As usual we introduce an origin  $O$  and consider the three location vectors

$$\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{OB}, \vec{c} = \overrightarrow{OC}.$$

From the sketch below we see that the difference vectors  $\vec{b} - \vec{a}$  and  $\vec{c} - \vec{a}$  lie in the desired plane. In consideration of the solution to Exercise b) we now determine a normal vector  $\vec{n}$  of the plane. As this vector must be perpendicular both to  $\vec{b} - \vec{a}$  and to  $\vec{c} - \vec{a}$ , we set

$$\vec{n} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}).$$

This brings us into the situation in Exercise b): the point  $R$  lies in the plane if and only if  $\vec{r} = \overrightarrow{OR}$  satisfies:

$$\vec{n}(\vec{r} - \vec{a}) = 0 \quad \text{or, written out,} \quad \left( (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) \right) (\vec{r} - \vec{a}) = 0.$$

d) Parametric representation of a plane

Exercise c) can also be approached in a different way. To this end, we introduce the shorthand  $\vec{p} = \vec{b} - \vec{a}$ ,  $\vec{q} = \vec{c} - \vec{a}$ . We observe immediately that for every point  $R$  in the plane  $E$  determined by  $A, B, C$ , the location vector  $\vec{r} = \overrightarrow{OR}$  can be written in the form

$$(**) \quad \vec{r} = \vec{a} + t\vec{p} + u\vec{q} \quad \text{mit } t, u \in \mathbb{R}$$

and, conversely, that every point which can be thus represented lies in the plane  $E$ .

With this we have found a further way of describing planes vectorially. In contrast to b) and c), this equation contains two real numbers, which as in a) are called parameters; the equation (\*\*\*) is known as the *parametric representation of the plane*. As a plane is a two-dimensional structure, it is natural that here — unlike in a) — two parameters should appear in its description.

**Important:**

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.