8. THE DERIVATIVE OF A VECTOR FUNCTION

(8.2) Vector functions

Preliminary remarks:

* The space $\mathbb{R}^3$ is sometimes challenging to work with - but we need it in the natural sciences.

* A map $t \to \vec{x}(t) \in \mathbb{R}^3$ can be interpreted in 2 ways:
a) dynamic: as the motion of a particle, or point mass, through space
b) static: as the parametric representation of a curve in $\mathbb{R}^3$.

* In (3.2) we considered a vehicle moving in a straight line, but possibly with some acceleration. In this setting we did not need to concern ourselves with magnitude and direction - we defined the velocity simply as the derivative of distance over time: $\dot{s} = v$. Now we consider velocity as a vector in $\mathbb{R}^3$. The magnitude of the vector is then the "speed", i.e. what is shown on the speedometer.

* * *

At the time t, the particle is located at the point R. Its position is thus defined by the vector

$$\vec{r} = \overrightarrow{OR} .$$

Since the point R and therefore the vector $\vec{r} = \overrightarrow{OR}$ change over time, instead of $\vec{r}$ we should really write

$$\vec{r}(t) .$$

So we are considering the vector $\vec{r}$ as a function of one variable (in our example, of the time t). This is called a vector-valued function or a *vector function*.

In order to be able to do calculations with these vector functions, we introduce a Cartesian coordinate system as in Chapter 2. With respect to this system, then, $\vec{r}(t)$ has the coordinates

$$\boxed{\vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.}$$

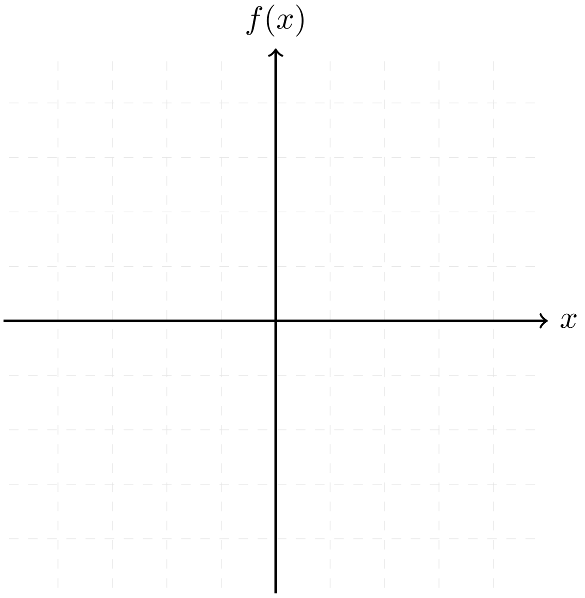
Along with $\vec{r}(t)$, the coordinates $x_i(t)$ $(i = 1, 2, 3)$ are dependent on the time t. Since $x_i(t)$ is always a real number, the *coordinate functions* (also referred to as component functions) $x_1(t), x_2(t), x_3(t)$ are just ordinary functions of one variable.

In this way, a vector function $\vec{r}(t)$ is expressed through three real-valued functions. Instead of $x_1(t), x_2(t), x_3(t)$ one can also write $x(t), y(t), z(t)$.

Beispiele

2. We consider the vector function

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}.$$



As the z–coordinate (3rd coordinate) here is equal to 0, the vector $\vec{r}(t)$ lies in the x-y–plane. (One could also just omit the 3rd coordinate; here we keep it, with a view to example 4 below!) Applying the familiar formula $\sin^2 t + \cos^2 t = 1$, we see that the points with coordinates $x = \cos t, y = \sin t$ must lie on the unit circle. It follows that the vector $\vec{r}(t)$ describes a *circular motion* (counterclockwise) of a particle in the x-y–plane. At time $t = 0$ the particle is located at the point (1,0), and again at time $t = 2\pi$ after it has completed a full revolution (and analogously at every point in time $t = 2n\pi, n \in \mathbb{Z}$).

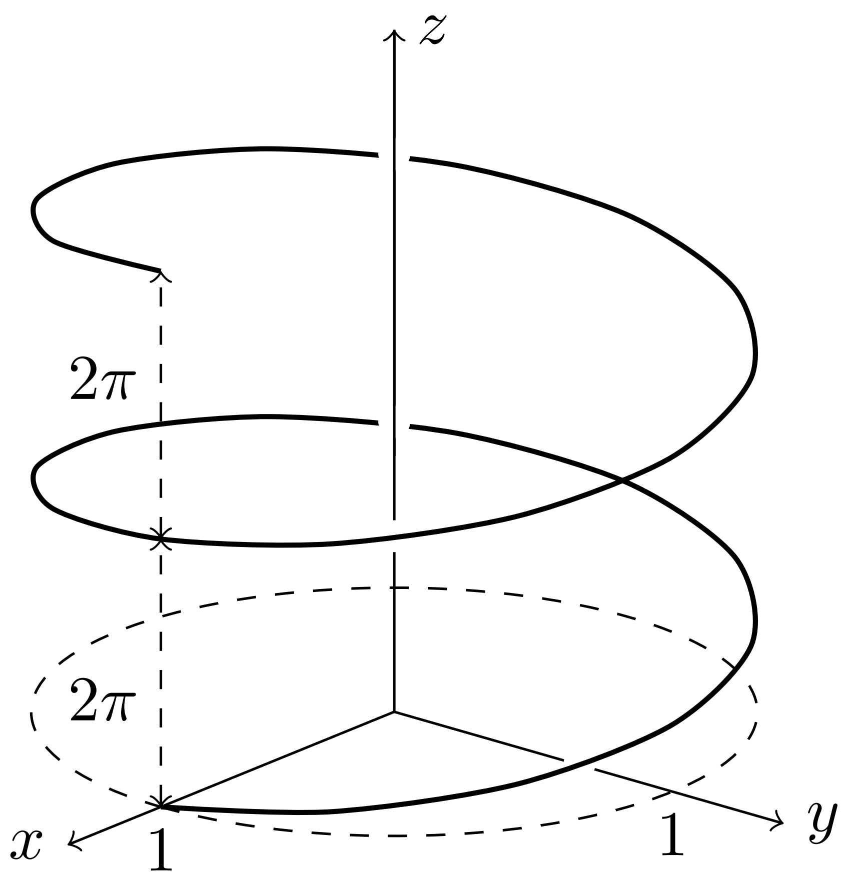
4. Finally, let
$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$
In contrast to Example 2 above, here the vector $\vec{r}(t)$ does not necessarily lie in the x-y–plane, but rather its endpoint at time t has a "height" (z–coordinate) of t.

The projection of $\vec{r}(t)$ onto the x-y–plane though is still the unit circle, exactly as in Example 2.

At time $t = 0$ we have $\vec{r}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

At time $t = 2\pi$ we have $\vec{r}(2\pi) = \begin{pmatrix} 1 \\ 0 \\ 2\pi \end{pmatrix}$.

From this we realise that $\vec{r}(t)$ describes a *helix* with "pitch" 2π and "period of revolution" 2π.



Short exercises:

An additional question:

(8.3) Parametric representations of curves

Sometimes we are interested less in the motion a function describes over time and more in the curve generated by its path as a geometric entity. In this case the variable t need not be interpreted as time, and is called a *"parameter"*.

$$\vec{u}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}, \quad t \in [0, 2\pi]$$

is according to this interpretation *one* parametric representation of a single revolution of the helix mentioned above.

In this setting (which, you will observe, is algebraically exactly the same as in (8.2)) we are not interested in the motion as such, but rather in the curve it generates, the "track" of the particle. This is then a "static" interpretation, in contrast to the "dynamic" interpretation in (8.2).

Thus in example 2 in (8.2), of course it makes a difference to the motion of such a particle whether it traverses the unit circle once or twice. The curve generated is the same either way though: the unit circle.

What is the difference between the following 3 formulations (all parametric representations of a line segment AB)?

$(*)$ $$\vec{u}(t) = \vec{a} + t\vec{c}, \quad t \in [0, 1]$$

$(**)$ $$\vec{v}(t) = \vec{a} + 2t\vec{c}, \quad t \in [0, \frac{1}{2}]$$

$(***)$ $$\vec{w}(t) = \vec{a} + (1-t)\vec{c}, \quad t \in [0, 1]$$

Here we return to the graphical representation of curves. With a little practice, it is possible to sketch the graph of the curve C that is generated by a given parametrisation. Naturally this exercise can also be delegated to an appropriate computer program.

Beispiele

1. Let

$$\vec{r}(t) = \begin{pmatrix} t \\ t^2 \\ 1-t \end{pmatrix}, \quad t \in [0,1]$$

First we consider its projection onto the x-y–plane. This is $x = x_1(t) = t,\ y = x_2(t) = t^2$, so $y = x^2$. So the projection of C onto the x-y–plane is a parabola. We see further that the projection of C onto the x-z–plane is a line, since

$x = x_1(t) = t, \quad z = x_3(t) = 1 - t.$

By first drawing in the parabola and the line, we are able to sketch C as well.

(8.4) The derivative of a vector function

We use the example of a particle's motion to motivate our definition. This movement is given by the "position function" $\vec{r}(t)$. We observe the values of this function at two points in time, t_0 and $t_0 + \Delta t$:

The difference $\Delta\vec{r} = \vec{r}(t_0 + \Delta t) - \vec{r}(t_0)$ is equal to the vector $\overrightarrow{PQ}$. During the time span Δt the particle has moved from P to Q (in general of course not in a straight line along the vector, but rather along some curved path). If we relate the change in position to the elapsed time, i.e. we divide by Δt, then we obtain

$$\frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t} .$$

This quantity is the *average velocity* (on the time interval $[t_0, t_0 + \Delta t]$). It consists of a *vector*, which shows the direction of average velocity and whose magnitude is a measure of the speed of the motion.

The expression $\frac{\Delta\vec{r}}{\Delta t}$ is nothing other than a vectorial *difference quotient* (cf. the analogue in (4.3.b)).

In order to obtain the instantaneous velocity (or simply: the *velocity*) at time t_0, we allow Δt to approach 0 as we did in (3.2), i.e. we take the limit

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t} .$$

This vector is naturally referred to as the *derivative* of the vector $\vec{r}(t)$ at the point t_0. In other words: velocity is defined as the derivative of the position function.

Letting go of this particular physical example, we make the following general definitions:

The vector function $\vec{r}(t)$ is said to be *differentiable* at the point t_0 if the limit

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}$$

exists. This limit is called the *derivative* of $\vec{r}(t)$ at the point t_0 and is denoted by

$$\frac{d\vec{r}}{dt}(t_0) \quad \text{or} \quad \vec{r}\,'(t_0) \quad \text{or} \quad \dot{\vec{r}}(t_0) \ .$$

It should be emphasised again: the derivative of a vector is another vector.

The two interpretations of $\dot{\vec{r}}(t_0)$:

<u>dynamic</u>

$\vec{r}(t)$ is the location vector
$\dot{\vec{r}}(t)$ is the velocity vector
$|\dot{\vec{r}}(t)|$ is the speed, i.e. as shown on the speedometer

<u>static</u>

$\vec{r}(t)$ is a parametric representation of a curve
$\dot{\vec{r}}(t_0)$ is the direction vector of the tangent to the curve at t_0

(8.5) Computing the derivative

To compute the derivative in practice, one uses the coordinate functions. Let

$$\vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Then we have

$$\dot{\vec{r}}(t) = \lim_{\Delta t \to 0} \begin{pmatrix} \dfrac{x_1(t+\Delta t) - x_1(t)}{\Delta t} \\ \dfrac{x_2(t+\Delta t) - x_2(t)}{\Delta t} \\ \dfrac{x_3(t+\Delta t) - x_3(t)}{\Delta t} \end{pmatrix} = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix}.$$

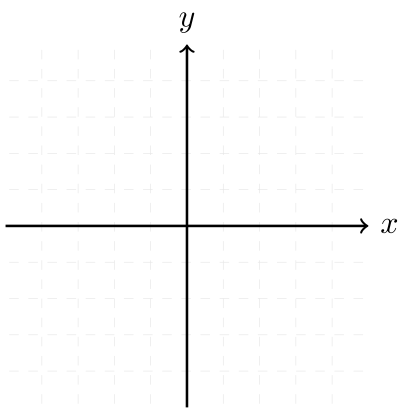
We have established:

> The derivative of a vector function
>
> $$\vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$
>
> is found coordinate-wise, i.e. it is
>
> $$\dot{\vec{r}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix}.$$

Beispiele

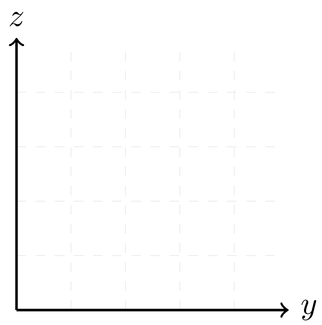
1. We analyse the helical motion

(♯) $$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}, \quad t \in \mathbb{R}$$



And now the speed (speedometer):

Does it make sense?



2. Let's vary this example a little. The same helix (as a geometrical shape) is described by

$$(\sharp\sharp) \qquad \vec{s}(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \\ 2t \end{pmatrix}, \quad t \in \mathbb{R} .$$

What is different now?

(8.6) Rules for the derivative of vector functions

Analogously to (5.2), the following rules hold for the derivative:

(1) Sum: $(\vec{u}(t) + \vec{v}(t))^{\cdot} = \dot{\vec{u}}(t) + \dot{\vec{v}}(t)$

(2) Difference: $(\vec{u}(t) - \vec{v}(t))^{\cdot} = \dot{\vec{u}}(t) - \dot{\vec{v}}(t)$

(3) Multiplication by a (constant) scalar: $(r\vec{u}(t))^{\cdot} = r\dot{\vec{u}}(t) \quad (r \in \mathbb{R})$

(4) Multiplicaion by a (scalar) function: $(r(t)\vec{u}(t))^{\cdot} = \dot{r}(t)\vec{u}(t) + r(t)\dot{\vec{u}}(t)$

(5) Scalar product of vectors: $(\vec{u}(t)\vec{v}(t))^{\cdot} = \dot{\vec{u}}(t)\vec{v}(t) + \vec{u}(t)\dot{\vec{v}}(t)$

(6) Vector product: $(\vec{u}(t) \times \vec{v}(t))^{\cdot} = \dot{\vec{u}}(t) \times \vec{v}(t) + \vec{u}(t) \times \dot{\vec{v}}(t)$

Observe that (4), (5), and (6) have exactly the same form as the usual product rule for real-valued functions. The anticommutativity of the vector product means that in (6) one must pay special attention to the order of the factors.

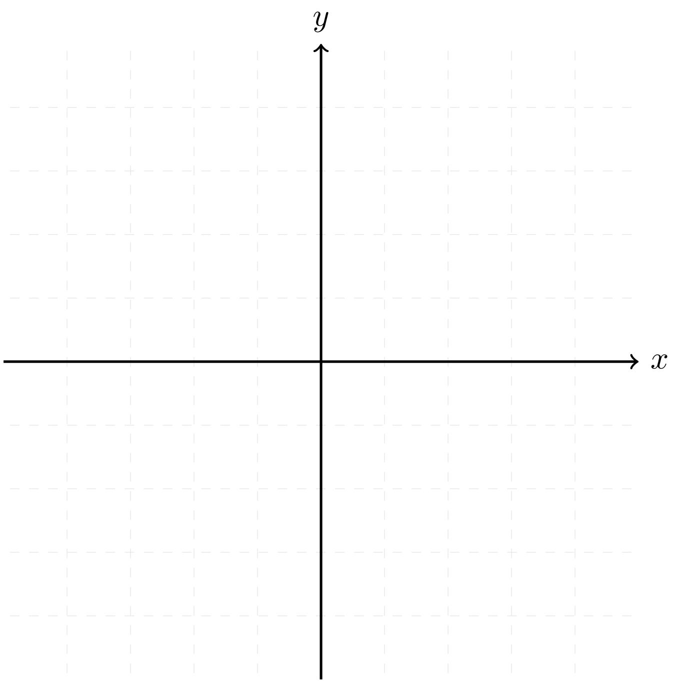
A short exercise: A particle moves according to

$$\vec{r}(t) = \begin{pmatrix} t \\ t^2 \\ (t-1)^2 \end{pmatrix}.$$

a) Sketch the path for $t \in [0, 1]$.
b) What is the magnitude of the velocity at times $t = 0$ and $t = 1$?
c) At what point in time is the speed minimal ($t \in [0, 1]$)? How big is it then?

Continued:

A trick for finding extrema of functions where the outer function is a square root, a logarithm, or the exponential function:



Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

The derivative of a vector function

$$\vec{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix}$$

is computed by differentiating the components, i.e. it is given by

$$\dot{\vec{r}}(t) = \begin{pmatrix} \dot{r}_1(t) \\ \dot{r}_2(t) \\ \dot{r}_3(t) \end{pmatrix}.$$

In analogy with (5.2) the follwing rules hold for the derivative:

(1) Sum: $(\vec{u}(t) + \vec{v}(t))^{\cdot} = \dot{\vec{u}}(t) + \dot{\vec{v}}(t)$

(2) Difference: $(\vec{u}(t) - \vec{v}(t))^{\cdot} = \dot{\vec{u}}(t) - \dot{\vec{v}}(t)$

(3) Product with a (constant) scalar: $(r\vec{u}(t))^{\cdot} = r\dot{\vec{u}}(t) \quad (r \in \mathbb{R})$

(4) Product with a (scalar) function: $(r(t)\vec{u}(t))^{\cdot} = \dot{r}(t)\vec{u}(t) + r(t)\dot{\vec{u}}(t)$

(5) Scalar product of vectors: $(\vec{u}(t)\vec{v}(t))^{\cdot} = \dot{\vec{u}}(t)\vec{v}(t) + \vec{u}(t)\dot{\vec{v}}(t)$

(6) Vector product: $(\vec{u}(t) \times \vec{v}(t))^{\cdot} = \dot{\vec{u}}(t) \times \vec{v}(t) + \vec{u}(t) \times \dot{\vec{v}}(t)$

Observe that (4), (5), and (6) have exactly the same form as the usual product rule for real-valued functions. The anticommutativity of the vector product means that in (6) one must pay special attention to the order of the factors.

A trick to help with finding extrema of functions - it might help if the outer function is the square root, logarithm, or exponential function: you can start by making a strictly monotonic transformation ($e^x, \ln(x)$ - depending on the domain of definition, possibly also square roots or polynomials)