4. THE DERIVATIVE (GERMAN: ABLEITUNG)

(4.2) The definition of the derivative (German: Ableitung)

Let $f : D(f) \to \mathbb{R}$ be a function (of one variable) with domain of definition $D(f)$ and let $x_0 \in D(f)$.

1) f is said to be *differentiable (German: differenzierbar) at the point* x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Among other things, this means: it cannot be $\pm\infty$. 0 is allowed.

2) f is said to be *differentiable on the subset* $X \subset D(f)$, if f is differentiable at every point $x_0 \in X$.

3) f is said to be *differentiable* (without further qualification), if this function is differentiable on its whole domain of definition.

4) If f is differentiable at the point x_0, then the number

$$\boxed{f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$

is called the *derivative* of f at the point x_0.

5) Let D' be the set of all points x at which f is differentiable. Then a new function

$$f' : D' \to \mathbb{R}, \quad x \mapsto f'(x)$$

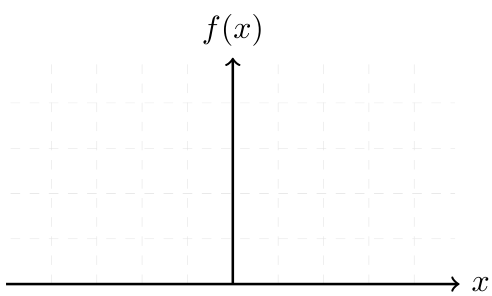
can be defined. This is called the *derived function* (or simply the derivative) of f.

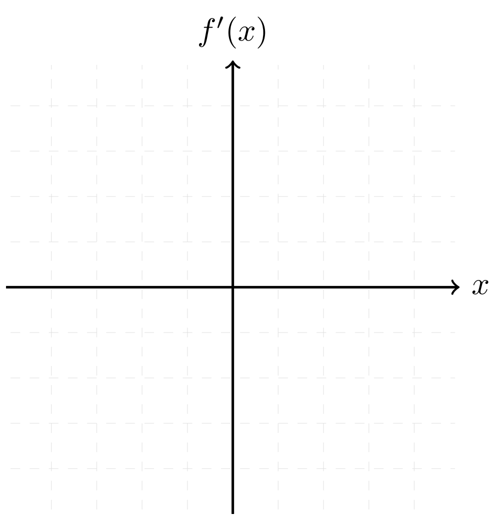
In the book you will find extensive comments on the details involved in this definition. It is very important to read through these! Many sources of misunderstanding are cleared up there.

(4.4) Examples illustrating differentiability

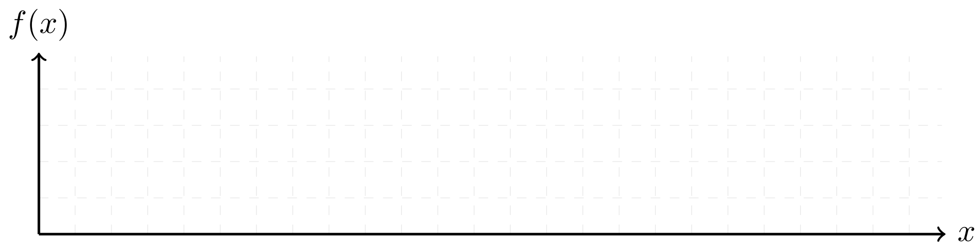
just a few examples, not for "all" functions, see Chapter 5 for more

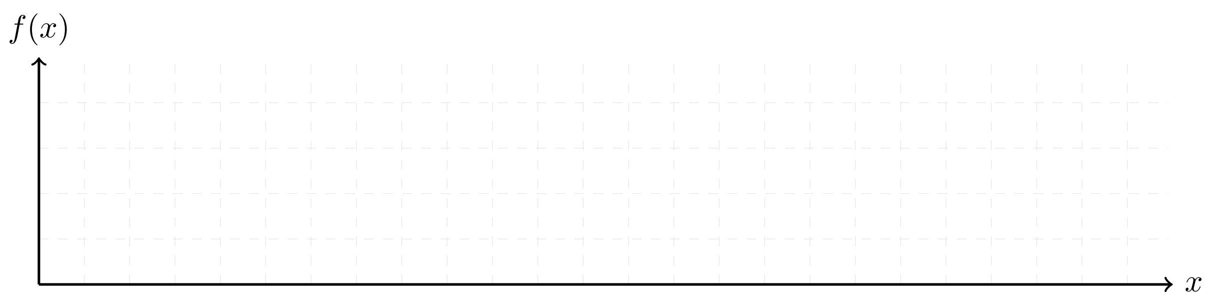
a) <u>The square function</u>

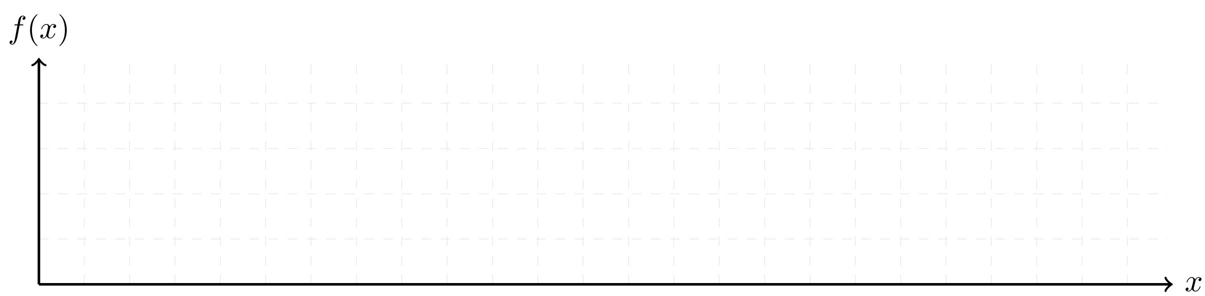




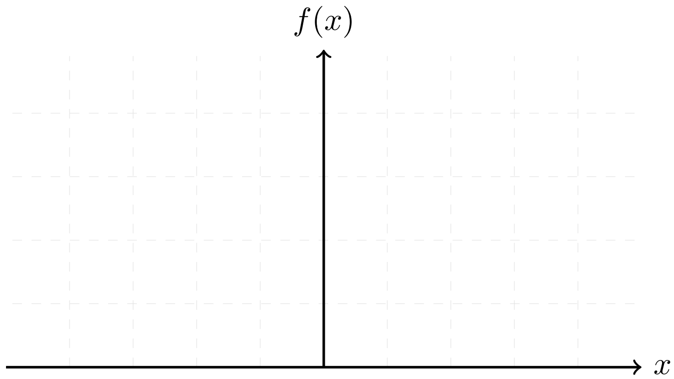
b) The square root function

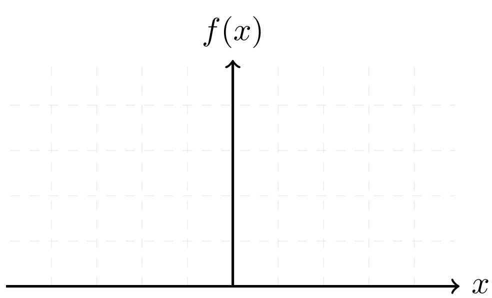


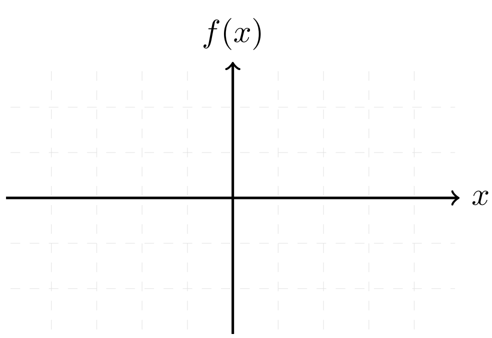




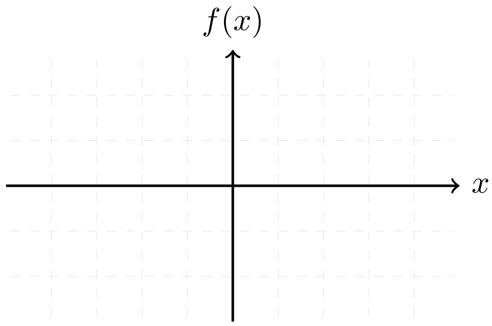
c) <u>The absolute value function</u>

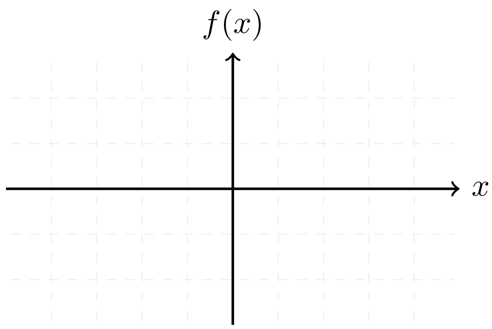


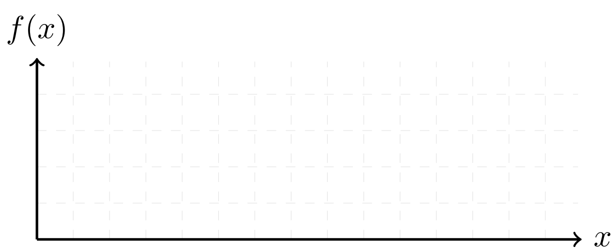




d) Nondifferentiability







(4.5) Higher derivatives

Let $f : D(f) \to \mathbb{R}$ be a function and D' as in (4.2.5) the set of all x at which f is differentiable. The *derived function*

$$f' : D' \to \mathbb{R}$$

is then defined on D'. (In most cases $D' = D(f)$, perhaps with the exception of a few points, cf. (4.4.b,c).)

If the function f' itself is differentiable at the point $x_0 \in D'$, then its derivative at this point is denoted by

$$f''(x_0)$$

and a new function can be defined, the *second derivative*

$$\boxed{f'' : D'' \to \mathbb{R}, \quad x \mapsto f''(x)}$$

(f' can of course also be referred to as the *first* derivative).

Continuing in this way, we can obtain the *higher derivatives*

$$f'' = f^{(2)},\ f''' = f^{(3)},\ f^{(4)}, \dots, f^{(n)}, \dots .$$

(It is impractical to write more than three apostrophes in a row.) In formulas it is sometimes convenient to set $f' = f^{(1)}$ and $f = f^{(0)}$; by the "zeroth derivative" we should thus understand the function itself.

The function f is said to be *infinitely differentiable* if $f^{(n)}$ exists for every $n \in \mathbb{N}$.

Other customary designations for the higher derivatives are:

$$\boxed{y''(x_0),\ \frac{d^2 f}{dx^2}(x_0),\ \frac{d^2 y}{dx^2}(x_0), \dots,\ \frac{d^n f}{dx^n} \quad \text{etc.}}$$

(Note the placement of the "exponents" in the last three expressions.)

In German: zweite Ableitung, dritte Ableitung, n-te Ableitung

Example

In (3.2) we have defined the instantaneous velocity $v(t)$ as

$$v(t) = s'(t) \ .$$

In exactly the same way, we can define the instantaneous rate of change in the velocity: this is given by

$$a(t) = v'(t)$$

and is of course familiar as the *acceleration* (at time t). If we combine these two equations, we find:

$$a(t) = v'(t) = s''(t) \ .$$

In physics it is customary to write derivatives with respect to time t using dot notation, as mentioned in (4.3.c). Thus:

$$a(t) = \dot{v}(t) = \ddot{s}(t) \ .$$

(4.6) Excursus: continuous functions

Intuitively, a continuous function means:

* a small change in x produces a small change (or no change at all) in y

* can be traced without lifting the pen (sufficient for this course)

* doesn't jump around

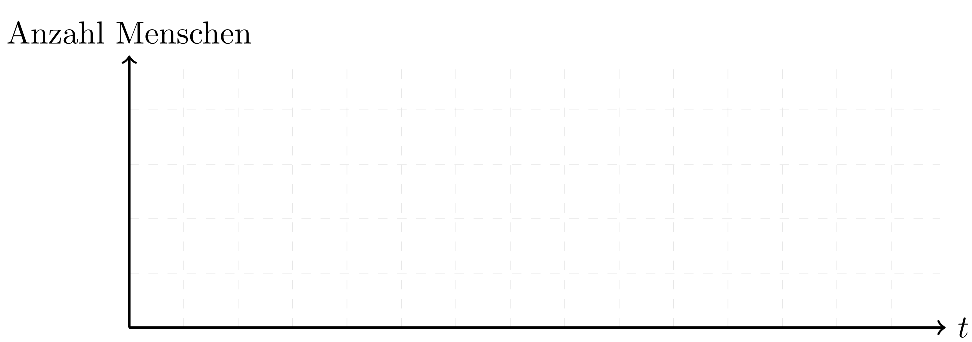
For example:

* a person's temperature as a function of time

* everyday air pressure; apart from explosions

* volume of a sphere as a function of the radius r: $V(r) = \frac{4}{3}\pi r^3$: volume increases continuously with r

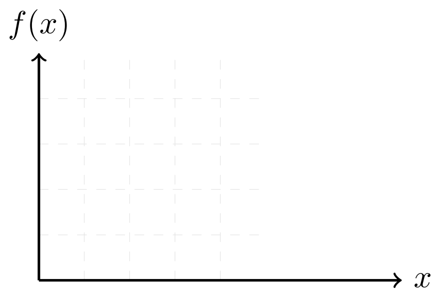
* population does not increase continuously: ± 1:

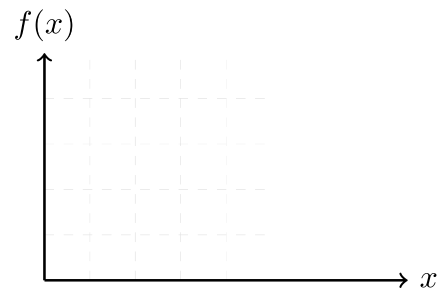


d) Definition

As demonstrated by the examples, continuity of a function means the following: the function f should be called "continuous at the point x_0" if $f(x)$ is close to $f(x_0)$ as long as x is sufficiently close to x_0. It is not yet quite clear though what "$f(x)$ is close to $f(x_0)$" should mean. Does the distance between $f(x)$ and $f(x_0)$ need to be less than 1/1000, less than 1/1,000,000, or less than something else? At this point it occurs to us that no prescribed number, no matter how small, will be satisfactory in every case, and therefore we arrive at the following definition:

The function f is *continuous (German: stetig) at the point* x_0 if $f(x)$ is arbitrarily close, i.e. as close as one could ever want it, to $f(x_0)$, provided that x is sufficiently close to x_0.

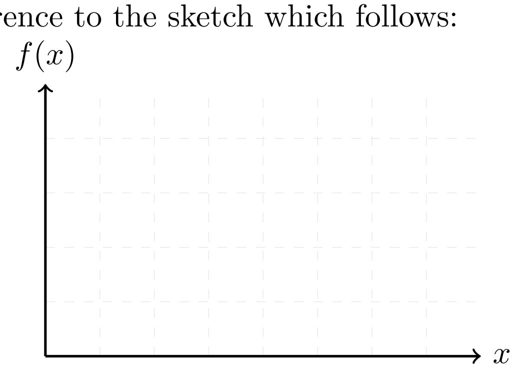




The definition above can also be expressed mathematically; we have done something quite similar already in (3.6b) for the precise definition of the limit:

Let $f : D \to \mathbb{R}$ be a function and let $x_0 \in D$. The function f is said to be continuous at the point x_0, as long as the following condition is satisfied: For every (even very small) number $\varepsilon > 0$, there exists a number $\delta > 0$ (depending on ε) such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in D$ with $|x - x_0| < \delta$. (The conditions that $x, x_0 \in D$ are of course understood.)

This condition for the continuity of f at x_0 can be understood graphically by reference to the sketch which follows:

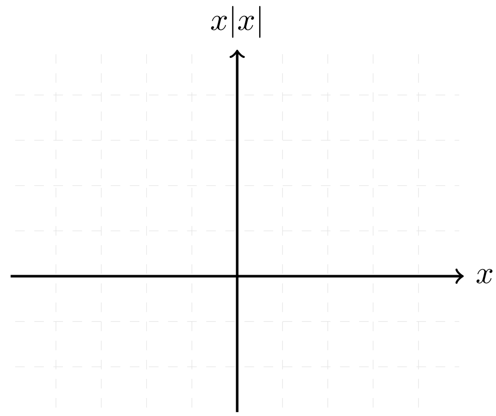


For every given "maximal deviation" $\varepsilon > 0$, then, a "tolerance" $\delta > 0$ can be given, such that for all x inside the region of this "tolerance" ($|x - x_0| < \delta$) $f(x)$ differs from $f(x_0)$ by less than the "maximal deviation" ($|f(x) - f(x_0)| < \varepsilon$).

e) Important continuous functions

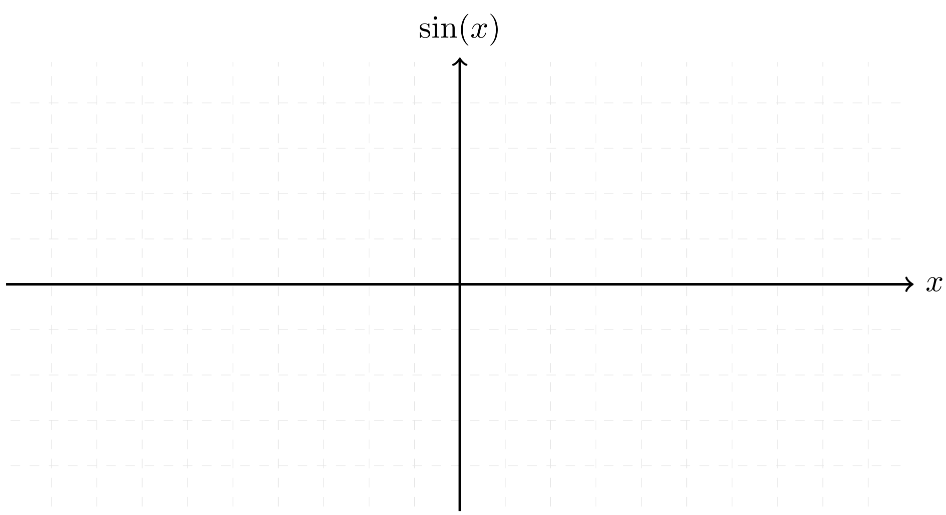
g) Continuity and differentiability

A 4-11

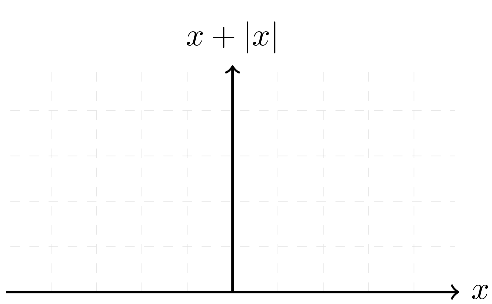


a)

b)



c)



How should we solve the exercises of Worksheets 4 and 5?

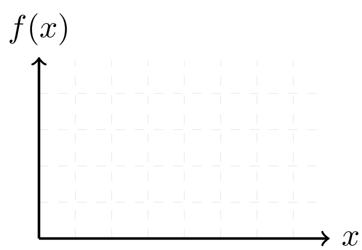
Where limits are to be computed:

In a course for mathematicians, we would use so-called ϵ-δ-arguments (see the small print in the book) to produce a thoroughly rigourous proof of a limit's value. In MAT 182 we will forgo this and proceed as follows:
1. We try to set x equal to x_0: $\lim_{x\to x_0} 2xx_0 = 2x_0^2$.
2. If this leads to 0/0 (as in the case of differential quotients) or to other problems, we will need to rearrange the expression, e.g. by polynomial division!

To check continuity and differentiability, e.g. in exercises involving piecewise defined functions ("curly braces"):

1. A function f is continuous at x_0 if the values as you approach x_0 from the left and from the right are the same, and if this is also equal to $f(x_0)$. Intuitively this means that the graph of the function does not have a hole or a jump at x_0.
2. A *continuous* (!) function is differentiable at x_0 if the values of its derivative in both regions similarly match as you approach x_0 from either side: in this case, the graph has no hole at x_0, no jump, and also no sharp corner.

Caution: Differentiability presupposes continuity! It can happen that a function has the same derivative from left and right, but still has a jump at x_0. This is why you should always check first whether the function is continuous at x_0.



Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

1) f is said to be *differentiable (German: differenzierbar) at the point* x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Among other things, this means: it cannot be $\pm\infty$. 0 is allowed.

2) f is said to be *differentiable on the subset* $X \subset D(f)$, if f is differentiable at every point $x_0 \in X$.

3) f is said to be *differentiable* (without further qualification), if this function is differentiable on its whole domain of definition.

4) If f is differentiable at the point x_0, then the number

$$\boxed{f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$

is called the *derivative* of f at the point x_0.

5) Let D' be the set of all points x at which f is differentiable. Then a new function

$$f' : D' \to \mathbb{R}, \quad x \mapsto f'(x)$$

can be defined. This is called the *derived function* (or simply the derivative) of f.

The function f is *continuous (German: stetig) at the point* x_0 if $f(x)$ is arbitrarily close, i.e. as close as one could ever want it, to $f(x_0)$, provided that x is sufficiently close to x_0.

If a function is differentiable, it must be continuous; the converse is not true! (absolute value function)

Caution: Differentiability presupposes continuity! It can happen that a function has the same derivative from left and right, but still has a jump at x_0. This is why you should always check first whether the function is continuous at x_0.