B. DIFFERENTIAL CALCULUS

3. MOTIVATING EXAMPLES FOR THE DERIVATIVE

(3.2) Velocity

If a car has traveled a distance of 150 km in two hours, then its average velocity during this time was evidently 75 km/h according to the formula

$$\text{average velocity } = \frac{\text{distance traveled}}{\text{time required}}\ .$$

See also https://schweizermonat.ch/stundenkilometer-und-andere-absurditaeten

(3.3) Rate of growth

Let $N = N(t)$ be the size of a bacterial culture, dependent on the time t. For two points in time t_0 and t_1 ($t_0 < t_1$) the *growth* during the interval $[t_0, t_1]$ is given by $\Delta N = N(t_1) - N(t_0)$. ($\Delta N$ may also be negative; a negative growth simply means a decline.)

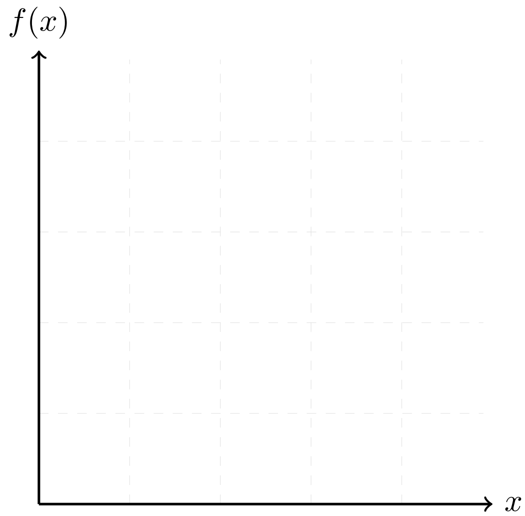
In order to obtain an impression of the speed of the culture's growth, one naturally considers the amount of growth ΔN in relation to the elapsed time $\Delta t = t_1 - t_0$, i.e. the length of the time interval under consideration. The expression

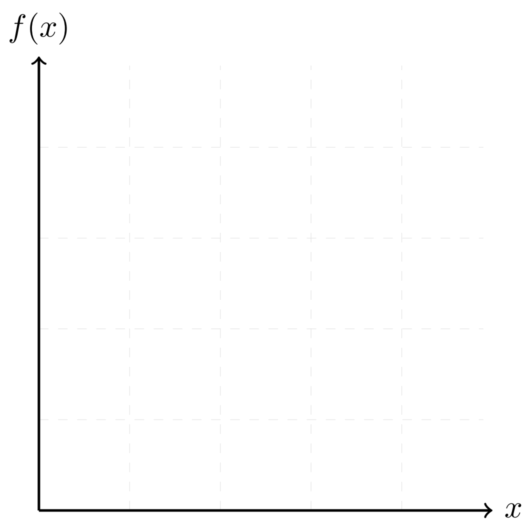
$$\frac{\Delta N}{\Delta t} = \frac{N(t_1) - N(t_0)}{t_1 - t_0}$$

represents the average growth per unit of time, also referred to as the "average growth rate".

(3.4) Tangent to a curve

We consider the graph of a function $y = f(x)$ and choose a value x_0 from its domain of definition. (Familiarity with these concepts is assumed; they are summarised in (26.9).) Now we want to determine the tangent to the curve at the point $P(x_0, y_0)$. Since this line must pass through P, it is enough to find the slope. To this end we pursue the following consideration, probably already familiar to you: we choose a value x_1, distinct from x_0, along the x–axis and consider the point $P_1(x_1, y_1)$ as well as $P(x_0, y_0)$. Here, x_1 may be either $> x_0$ or $< x_0$.





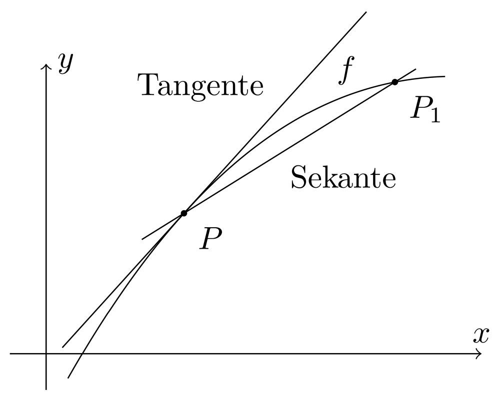
With $\Delta f := f(x_1) - f(x_0)$ and $\Delta x := x_1 - x_0$, the slope $a = \tan\varphi_1$ of this secant is given by

$$a = \frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \; .$$

As we allow x_1 to approach x_0, the secant becomes closer and closer to that line which we intuitively regard as the tangent.

For the slope $\tan\varphi$ of the tangent, then, one reasonably arrives at the following definition:

$$\tan\varphi = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \; .$$



Once again, the limit has the same form as in (3.2).

It should also be noted that x can approach x_0 equally from below (from the left) or from above (from the right). If we are to speak sensibly about "the" slope of the tangent at x_0, then in both cases the same value $\tan\varphi$ of the slope must be obtained (cf. also (4.4.c)).

Here we give a simple numerical example for the function $f(x) = x^2$ and the point $x_0 = 1$. In particular, the last column gives the slope of the secant.

x	x_0	$\Delta x = x - x_0$	$f(x) = x^2$	$f(x_0) = x_0^2$	$\Delta f = f(x) - f(x_0)$	$\frac{\Delta f}{\Delta x}$
2	1	1	4	1	3	3
1.5	1	0.5	2.25	1	1.25	2.5
1.3	1	0.3	1.69	1	0.69	2.3
1.1	1	0.1	1.21	1	0.21	2.1
1.05	1	0.05	1.1025	1	0.1025	2.05
1.01	1	0.01	1.0201	1	0.0201	2.01
1.001	1	0.001	1.002001	1	0.002001	2.001

(3.6) Limits of functions

a) <u>Examples</u>

b) <u>The definition</u>

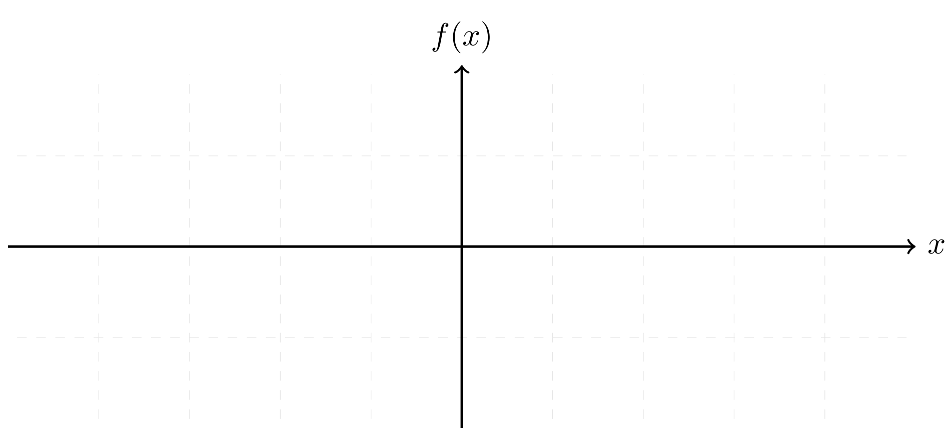
On the basis of the informal examples above, we state the following:

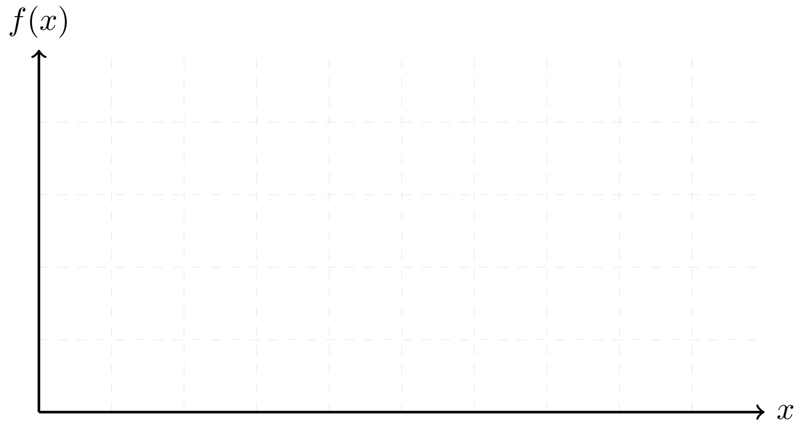
The number r is called the *limit* of the function g at the point x_0, if $g(x)$ is arbitrarily close to r, for all values of x ($x > x_0$ and $x < x_0$) which are sufficiently close to x_0 although always with $x \neq x_0$. This is written using the following notation:

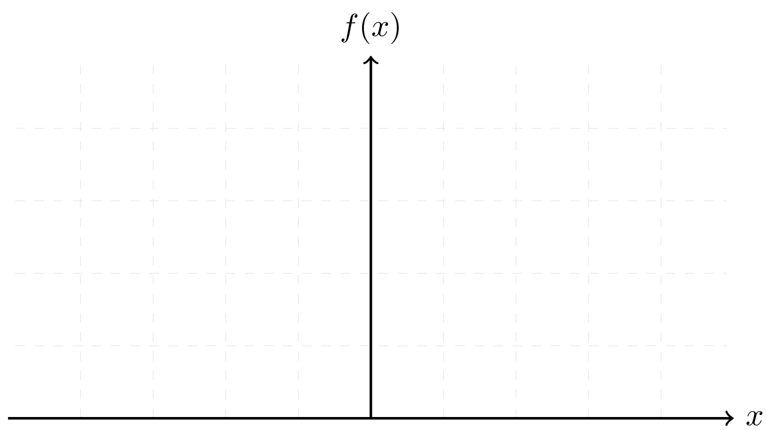
$$\lim_{x \to x_0} g(x) = r \quad \text{or} \quad g(x) \to r \ \text{ for } \ x \to x_0 \ .$$

Noteworthy here is that $\lim\limits_{x \to x_0} g(x)$ is determined only by the surrounding values $g(x)$ for $x \neq x_0$. The function g does not even need to be defined at x_0.

c)-e) <u>A few notes on limits, further definitions, and examples</u>







Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

The differential quotient is

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \ .$$

The derivative is then

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \ .$$