20. IMPROPER INTEGRALS

German: uneigentliche Integrale

Preparation: how can we decide: $\lim_{x\to\infty} f(x)g(x) =?$

A few mathematical rules; if $x \to \infty, c > 0$ is a constant; how fast do $x!$, e^x , x^c , $\ln(x)$ grow?

$$\lim_{x\to\infty} \frac{e^x}{x^{1000}} =$$

$$\lim_{x\to-\infty} xe^x =$$

As an aside: from L'Hôpital's Rule for $x > 0$: $\lim_{x\to 0} x\ln(x) =$

First type of problem: we are integrating over an infinitely long interval

until now: $\int_a^b f(x)dx$, where f is continuous and a, b are fixed and finite.

now: $\int_a^\infty f(x)dx$, $\int_{-\infty}^b f(x)dx$, $\int_{-\infty}^\infty f(x)dx$ - how does that work?

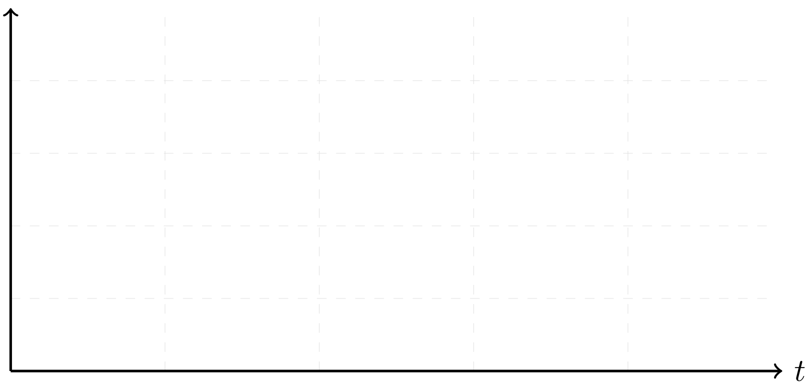
We *define* (!) : $\displaystyle\int_a^\infty f(x)dx :=$

And then what about an analogue to the left? $\displaystyle\int_{-\infty}^b f(x)dx :=$

These definitions make sense intuitively and mathematically.

Example from physics (time until an isotope decays, density function, discussed here verbally, more in MAT 183 and in a physics course):

$f(t) = \lambda e^{-\lambda t}$



And now even from $-\infty$ to $+\infty$: $\int_{-\infty}^{\infty} f(x)dx$ - how should we define that?

3 pages from now we will calculate $\int_{-\infty}^{\infty} x\,dx$ - does this maybe yield 0 for reasons of symmetry?

From earlier: if a and b are finite, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

for all $c \in \mathbb{R}$, typically $c \in [a, b]$.

Now for the crucial definition:

If for every choice of c the two integrals

$$\int_{-\infty}^{c} f(x)dx \text{ and } \int_{c}^{\infty} f(x)dx$$

both exist - that is, are in the interval $(-\infty, +\infty)$, neither $-\infty$ nor $+\infty$, actually finite,

Then we say (cf. a mathematics course):

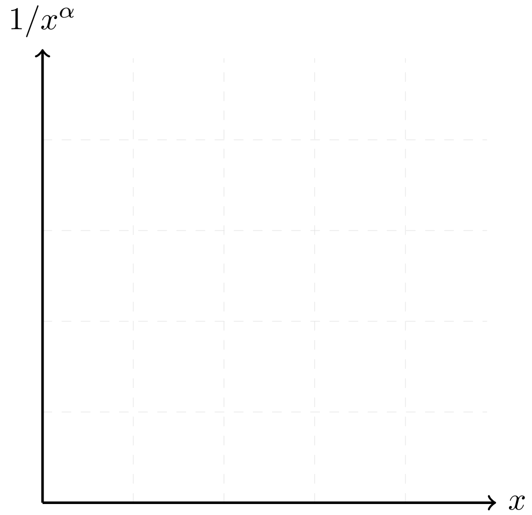
$$\int_{-\infty}^{\infty} f(x)dx := \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

for arbitrary c - the value is always the same.

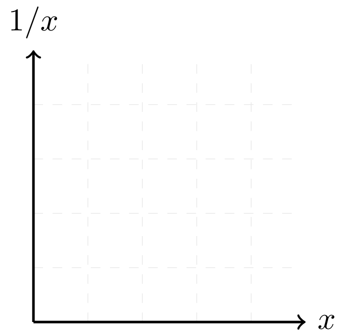
Otherwise we say that the integral is undefined.

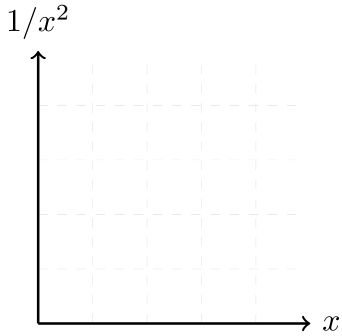
Second type of problem: the value of the function itself approaches ∞

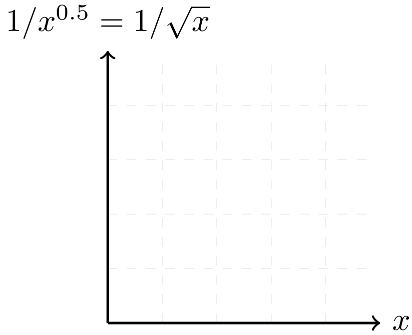
Prelude: what's the correct way to draw hyperbolae $f(x) = \frac{1}{x^\alpha}$?



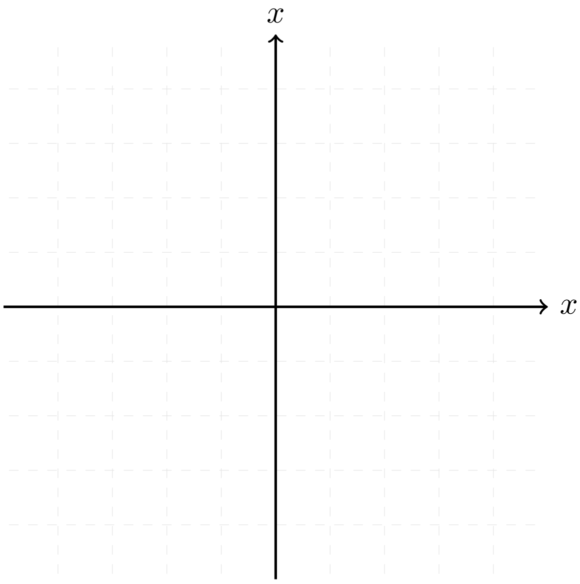
A rule of thumb for (almost) all cases:







$$\int_{-\infty}^{\infty} x\,dx = 0?$$



Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

First type of problem (an infinitely long interval):

$$\int_a^\infty f(x)dx := \lim_{t\to\infty} \int_a^t f(x)dx$$
$$\int_{-\infty}^a f(x)dx := \lim_{t\to-\infty} \int_t^a f(x)dx$$

If for every choice of c the two integrals

$$\int_{-\infty}^c f(x)dx \text{ and } \int_c^\infty f(x)dx$$

both exist - that is, are in the interval $(-\infty, +\infty)$, neither $-\infty$ nor $+\infty$, actually finite,

Then we say (cf. a mathematics course):

$$\int_{-\infty}^\infty f(x)dx := \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

for arbitrary c - the value is always the same.

Otherwise we say that the integral is undefined.

Second type of problem (poles of the function; i.e. function value approaches ∞):

If for example there is a pole at 0:

$$\int_0^a f(x)dx := \lim_{t\searrow 0} \int_t^a f(x)dx$$