2. VECTOR ARITHMETIC WITH COORDINATES

(2.2) The Cartesian coordinate system

We choose an origin O in space and a rectangular (instead of "rectangular" this is also referred to as "Cartesian", after R. Descartes (in Latin, Cartesius) (1596–1650)) coordinate system with an x–axis, y–axis and z–axis, which are arranged in the usual ordering:

$\boxed{\text{(2.3) The coordinates of a vector}}$

(2.4) Arithmetic operations with coordinates

In Chapter 1 we defined the arithmetic operations for vectors in a geometric way. How can the coordinates be used to perform these operations computationally? In the following table we assemble the answer.

Let the vectors

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

be given. Then

(1) $$\vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix},$$

(2) $$\vec{a} - \vec{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix},$$

(3) $$r\vec{a} = \begin{pmatrix} ra_1 \\ ra_2 \\ ra_3 \end{pmatrix},$$

(4) $$\vec{a}\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \,,$$

(5) $$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

In addition the following formulae hold:

(6) The length of a vector $\vec{a}$ is given by

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\,.$$

(7) The angle φ between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\varphi = \frac{\vec{a}\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\,.$$

Notes / Clarifications:

(2.5) Proofs for a few of these rules

(2.6) Length of a vector in all dimensions; equation of a circle/sphere en passant: equation of a circle/sphere

(2.7) A few examples

Distance, angle

Area of a triangle, parametric equation of a line

Equation of a plane

Finally: www.youtube.com/watch?v=gfWpNCFjgfY

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

Aside from a few trivial rules, it is also true that:

(4)
$$\vec{a}\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \ ,$$

(5)
$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} .$$

(6) The length of a vector $\vec{a}$ is given by

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \ .$$

(7) The angle φ between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\varphi = \frac{\vec{a}\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}} \ .$$

Equation of a circle: $r^2 = (x-a)^2 + (y-b)^2$

Equation of a sphere: $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

Area of a triangle can be computed as length of the vector product divided by 2.

The normal vector to a plane:

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Longleftrightarrow ax + by + cz = d.$$