After this, we will first discuss Chapters 22 and 23 and then (time permitting) return to Chapter 19; please bring the appropriate material to class.

18. SOME IMPORTANT FUNCTIONS AND THEIR APPLICATIONS

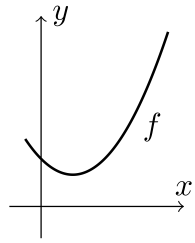
(18.2) Modification of a function

Many dependencies in nature and technology are not described by functions in their simplest form, i.e. not by such functions as

$$x^2, e^x, \sin(x)$$

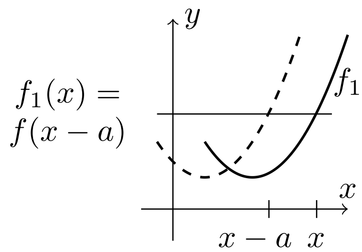
and so on. Rather, these functions typically have to be adjusted to fit the reality. The following modifications to functions are something you will need to have at your fingertips in your field of study. They are also frequently encountered in Exercise 1 on the exam!

We consider a function $f(x)$ and its graph, which is given by the relation $y = f(x)$. Now we modify this relation in various ways and see what results.

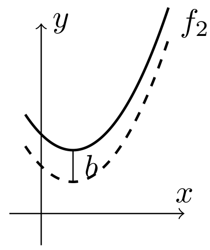


a) Shift in the x–direction

We let $f_1(x) = f(x-a)$ for $a > 0$. Compared to the graph of f, the graph of f_1 is shifted to the right by a units. Similarly, $f(x+a)$, $a > 0$ yields a shift to the left.

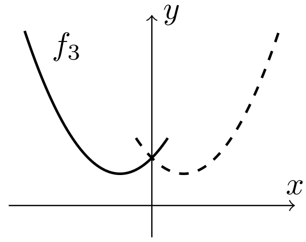


b) Shift in the y–direction



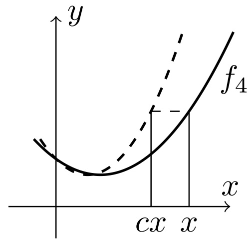
Now let $f_2(x) = f(x) + b$ for $b > 0$. Here the graph of f is shifted upward by b units. Analogously, $f(x)-b,\ b > 0$ results in a shift downward.

c) Reflection across the y–axis



The graph of the function $f_3(x) = f(-x)$ is obtained by reflecting f across the y–axis.

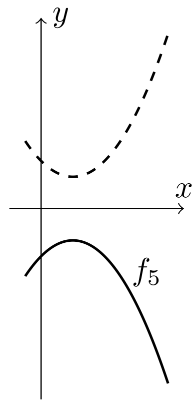
d) Stretching/compression in the x–direction



Let $c > 0$. We consider $f_4(x) = f(cx)$. The value of f_4 at the point x is equal to that of f at the point cx. Thus the transformation of f to f_4 results in a stretching in the x–direction by a factor of $\frac{1}{c}$. (For $c > 1$, i.e. $\frac{1}{c} < 1$, this corresponds intuitively to a “compression”.)

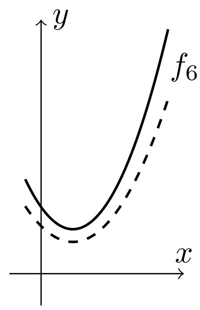
If $c < 0$, then in accordance with c) we have a scaling by $\frac{1}{|c|}$ combined with a reflection across the y–axis.

e) Reflection across the x–axis



This transformation is achieved by $f_5(x) = -f(x)$.

f) Stretching/compression in the y–direction



Let $s > 0$. We consider $f_6(x) = sf(x)$. The value of f_6 at the point x is s times the value of $f(x)$. The transformation of f to f_6 consists of a stretching ($s > 1$) (or a compression ($s < 1$)) in the y–direction.

For $s < 0$ this is combined with a reflection across the x-axis as in e).

These modifications can naturally also be combined; in the extreme case this leads to

$$g(x) = sf\big(c(x-a)\big) + b\ .$$

This topic is taken up in (18.3).

Let's quickly take a more precise look at cases a) and d) which are particularly mistake-prone:

a) For $a > 0$, for example $a = 2$ (as a general didactic tip: when you don't understand something, **plug in concrete numbers**), one might see $f_1(x) := f(x - a)$ and think that the "$-a$" must mean it moves a units to the left ("smaller", minus sign) gehen muss. The correct interpretation: you have to go $a = 2$ units to the right in order to get the same values as before, because of course $a = 2$ is immediately subtracted:

d)

One last didactic note: always check these things by using positive (i.e. > 0) a, c, ideally all > 1, in the 1st quadrant.

(18.3) Periodic functions

Why we need trigonometric functions:

- for geometry (triangles) as introduced in high school.
- unexpectedly: for integration (see chapter 17).
- highly mathematically: because they are periodic, and therefore building blocks for all "reasonable" periodic functions, see immediately below.

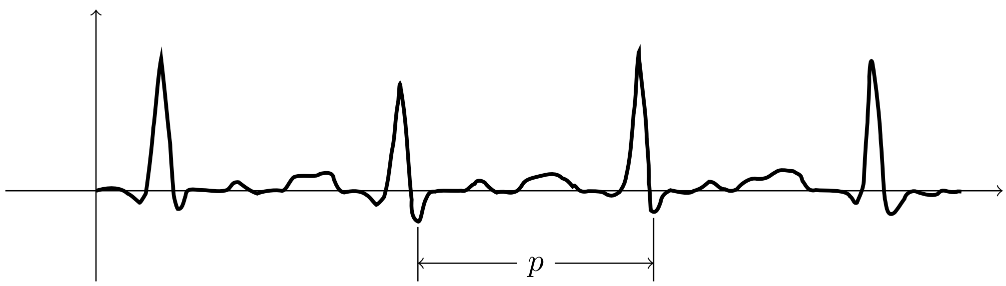
It is well known that for all $x \in \mathbb{R}$ (here we use radians as usual)

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x \ .$$

In general: a function f is said to be *periodic* if there exists a number $p > 0$ such that $f(x + p) = f(x)$ for all x.

For the sine function one can take $p = 2\pi$, but $p = 4\pi, 6\pi$, etc. would also be possible. The smallest positive number p with the property mentioned above is called the *period* of f (so in the case of the sine function, $p = 2\pi$).

Periodic phenomena are quite frequently encountered in nature (one can think about oscillations, biorhythms, and so on). This means that periodic functions arise in a very natural way. As a concrete example we mention the electrocardiogram (ECG), which (with a certain amount of idealisation) can be regarded as the graph of a periodic function:



At first this curve would appear to have nothing at all to do with any trigonometric functions. There is however an important mathematical theorem which says that every periodic function f which is in some sense "reasonable" can be represented as a so-called *Fourier series*, namely an infinite series of the form

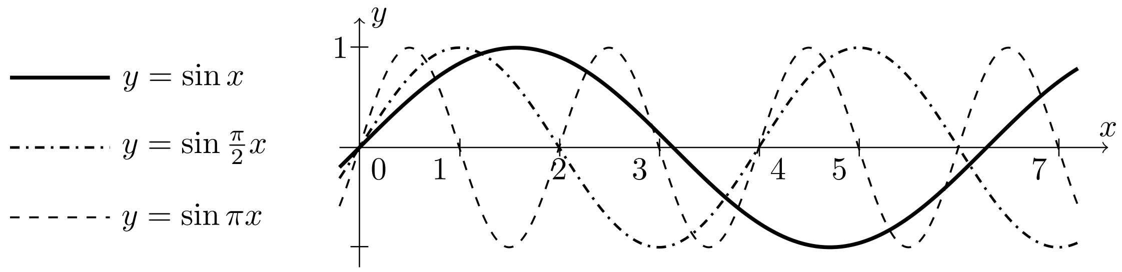
$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ .$$

(Even the function given by the above ECG trace, for example, would be "reasonable" enough!) This formulation presupposes that f has period 2π, which is easy to achieve by scaling (see below).

We will not be able to examine these Fourier series in more detail here. They are mentioned in order to show that the significance of trigonometric functions extends far beyond calculations about triangles, as according to the above observations they form a set of building blocks, so to speak, for periodic functions in general.

Period p and frequency f:

From 2π to an arbitrary period p:



The sine function is compressed more and more; the y-values repeat themselves sooner. What is the new period?

Proof:

Some calculations to check the above:

Parallel shift:

New amplitude:

Example (a possible exam question); try to find a periodic function h with the following properties:

- period 48 (hours),
- a zero (where the function is increasing) at $x = 12$,
- amplitude 6.

Wavelength vs. period: Wavelength is the length of the wave in meters; period is the time per oscillation in seconds!

At this point in the textbook there are discussions of the exponential function, radioactive decay, and C14 dating method – please read these again **thoroughly**. After that come the hyperbolic functions and their inverses (German: Area-Funktionen) – please at least **read through these once**. Afterward is some important, easily digestible material in (18.9); please read this well too.

(18.10) Logarithmic scales

Logarithmic scales are important for applications in the natural sciences - they are covered in great detail in the textbook. The most important points follow here. Logarithm paper can be printed from

http://de.wikipedia.org/wiki/Logarithmenpapier

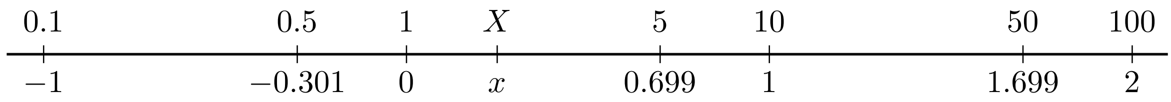
1. First a few observations on notation, terminology, motivation:

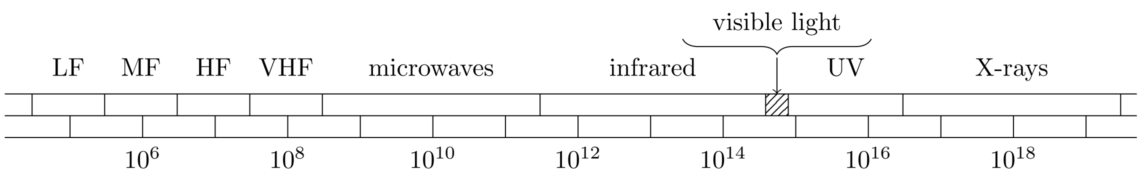
"*semi*-logarithmic" is the same thing as "*simply* logarithmic" (a case of historical error in the naming convention, sorry); contrast with *doubly* logarithmic.

Why log scales:
1. In the natural sciences we often have measurements which by their nature are greater than 0 (weights, lengths) and which also span an enormous range (see frequencies, next page). The logarithm is defined for numbers > 0 and a logarithmic scale lets us represent the whole range quite compactly (the physics stays the same - it's only a question of graphical representation).
2. If we have a 2-dimensional functional relation, we can use the right transformation to represent an exponential or power relation as linear (again, the physics doesn't change).

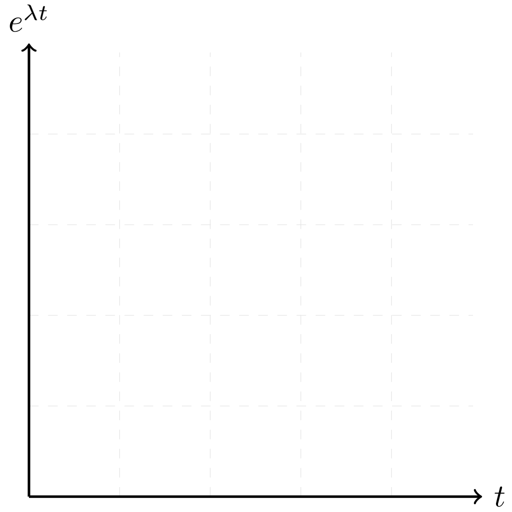
Side note: as we will learn in MAT 183, logarithmic transformations are almost only used with so-called ratio scales.

2. One-dimensional

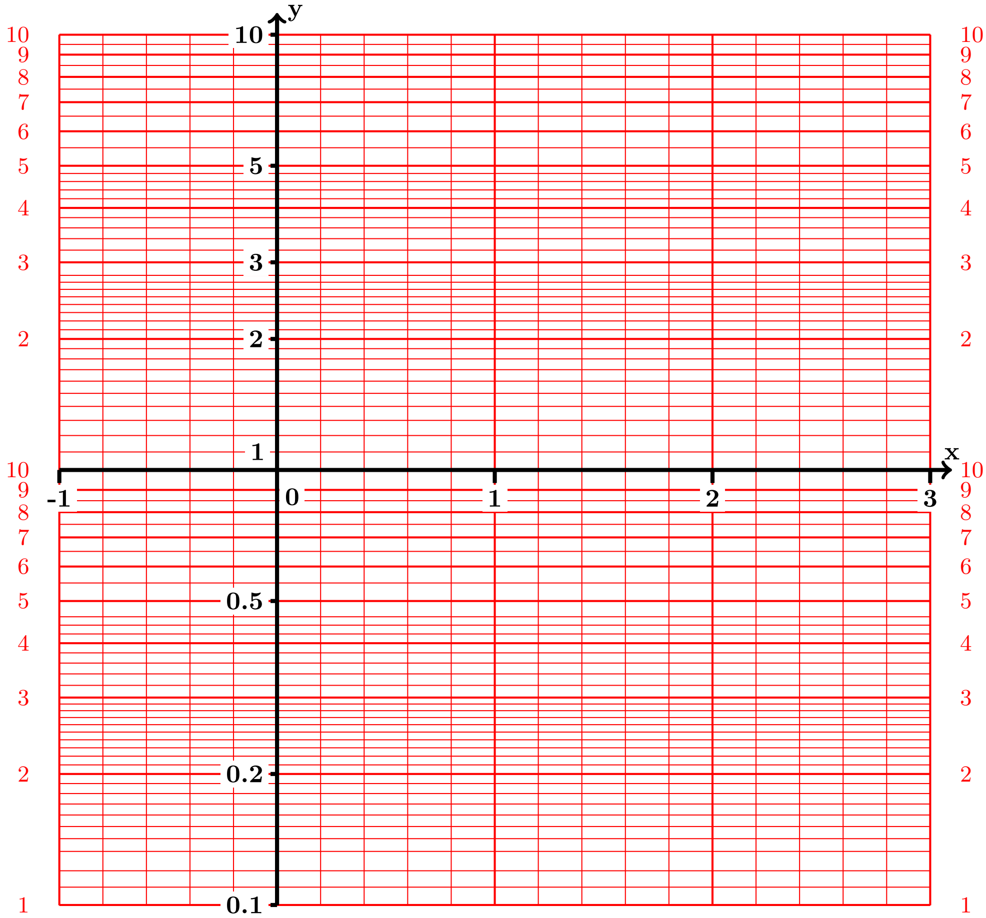




3. Two-dimensional I: exponential (or geometric) growth:



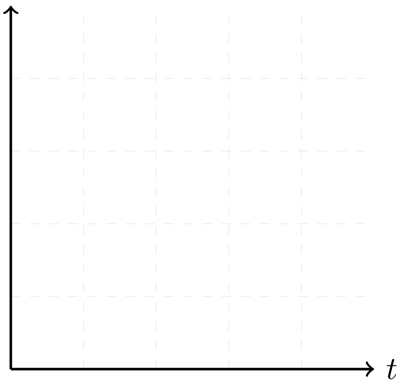
Logarithms of y-values are plotted against the original x-values (here t-values)
$\Rightarrow$ exponential growth becomes *in this graphical representation* (!) a straight line.


y
x
10
5
3
2
1
0.5
0.2
0.1
-1
0
1
2
3

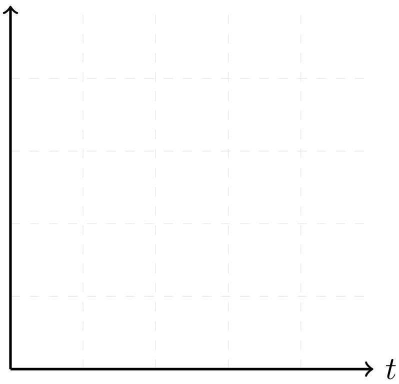
Related: The 2020 coronavirus outbreak: the small subtle differences...

Chapter 15: exponential growth as soon as $R_0 > 1$:

$y(t) = e^{\lambda t}$

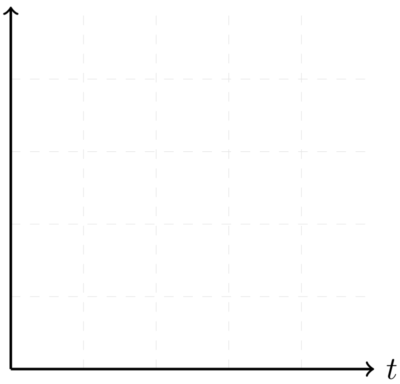


$\ln(y(t)) = \lambda t$

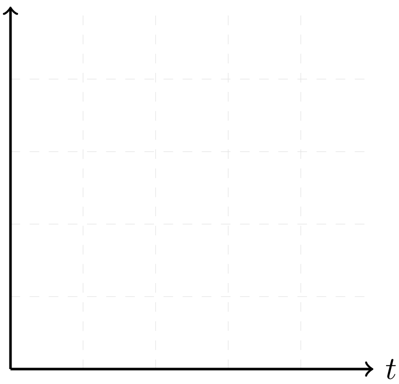


Summer 2020:

$y(t) = e^{\lambda t}$

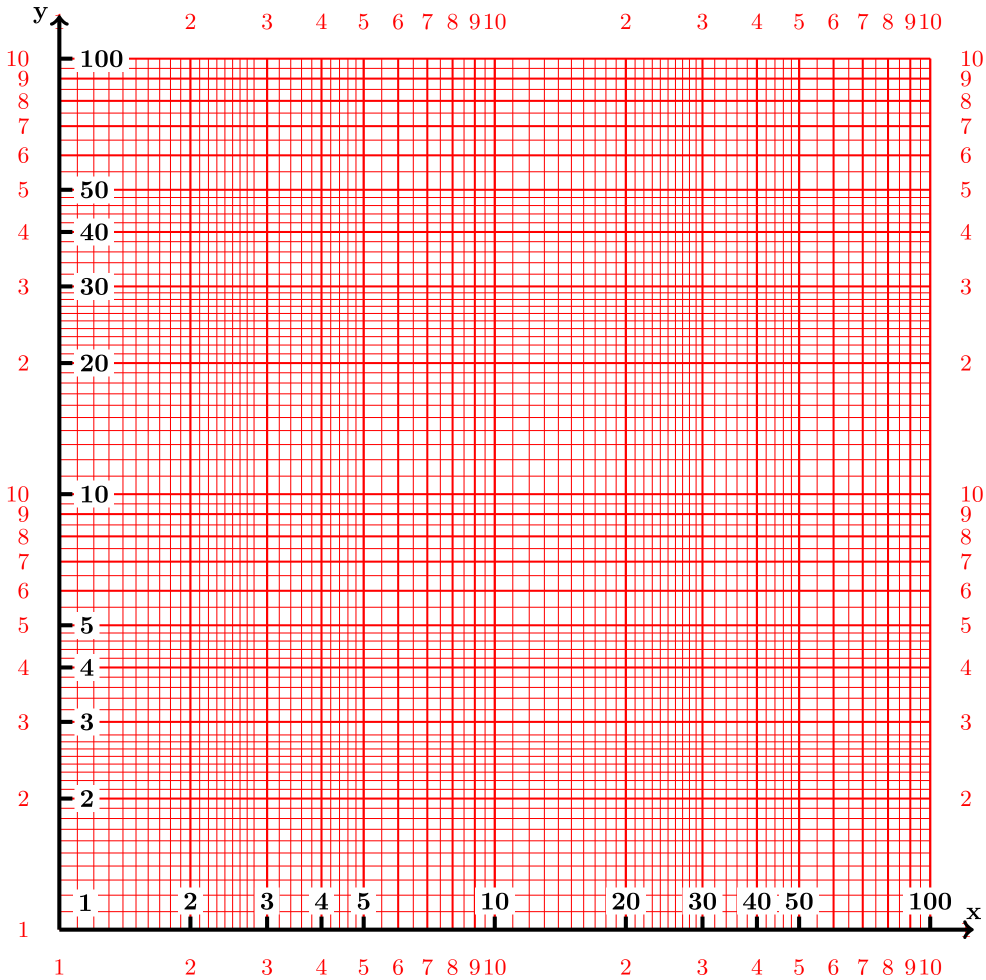


$\ln(y(t)) = \lambda t$



4. Two-dimensional II: power relationships:

Logarithms of y-values are plotted against logarithms of x-values
$\Rightarrow$ a power relationship becomes *in this graphical representation* (!) a straight line.



An example with real data:

https://schweizermonat.ch/die-berechnung-der-langsamkeit/

5. Concluding remarks

The inverse of all these relationships is also true:

- simple log (y-values logarithmic) and a line $\Leftrightarrow y = Ke^{\lambda t}$ (exponential)
- double log and a line $\Leftrightarrow y = Kx^n$ (power relationship)

Didactic tip: if you make data transformations (e.g. log) you should justify them: "Since we have exponential growth, I plot the y-values logarithmically."

General notes on graphics in presentations (not only when using logarithmic scales)

- please label the axes, including origin and units
- when possible avoid 3D graphs
- would a table not be better?

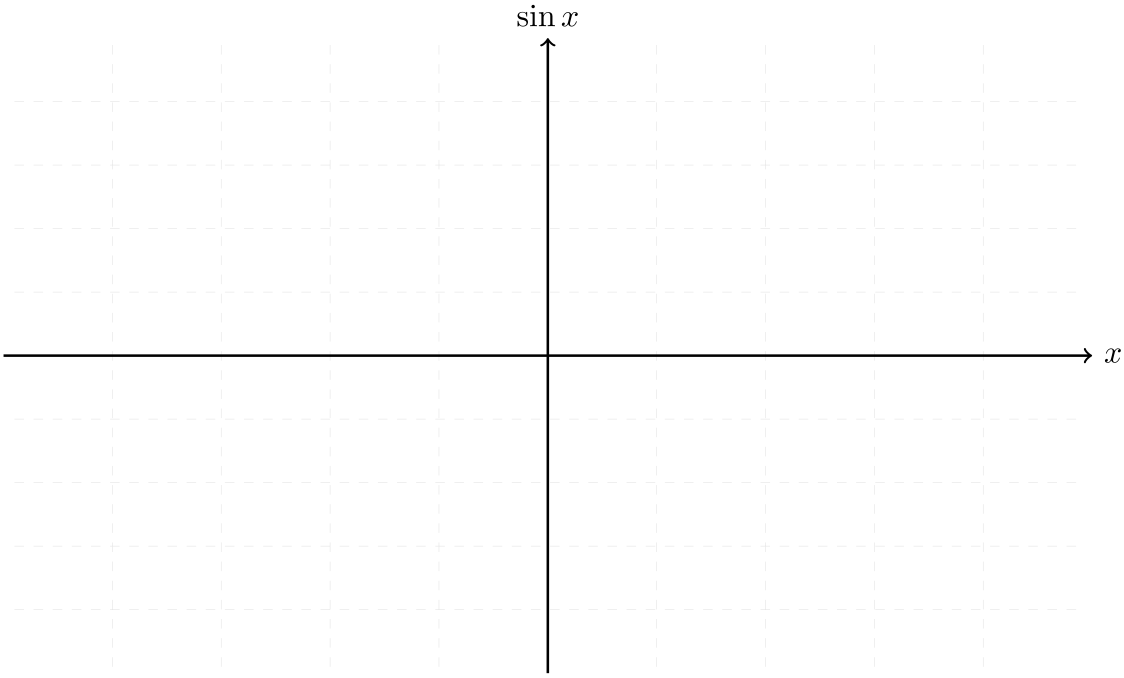
Then: give the audience plenty of time to orient themselves, help them with narration like this: the x-axis is..., the y-axis describes / this region / if all the points were on a line at 45 degrees, it would mean that / the extreme point here means that ...., this other extreme point / as a 22-year-old student with parental income of ... you would wind up in 20 years somewhere around here (emotions increase attention) / and here you see the last numerical example represented by this point / these two points have the same x-value but completely different y-values. This is probably because.... If you use logarithmic scales allow plenty of time; especially because of values which are 0 or negative.

More on this at www.acad.jobs/docs/vortragstipps.html.

Important examples of logarithmic / exponential relationships

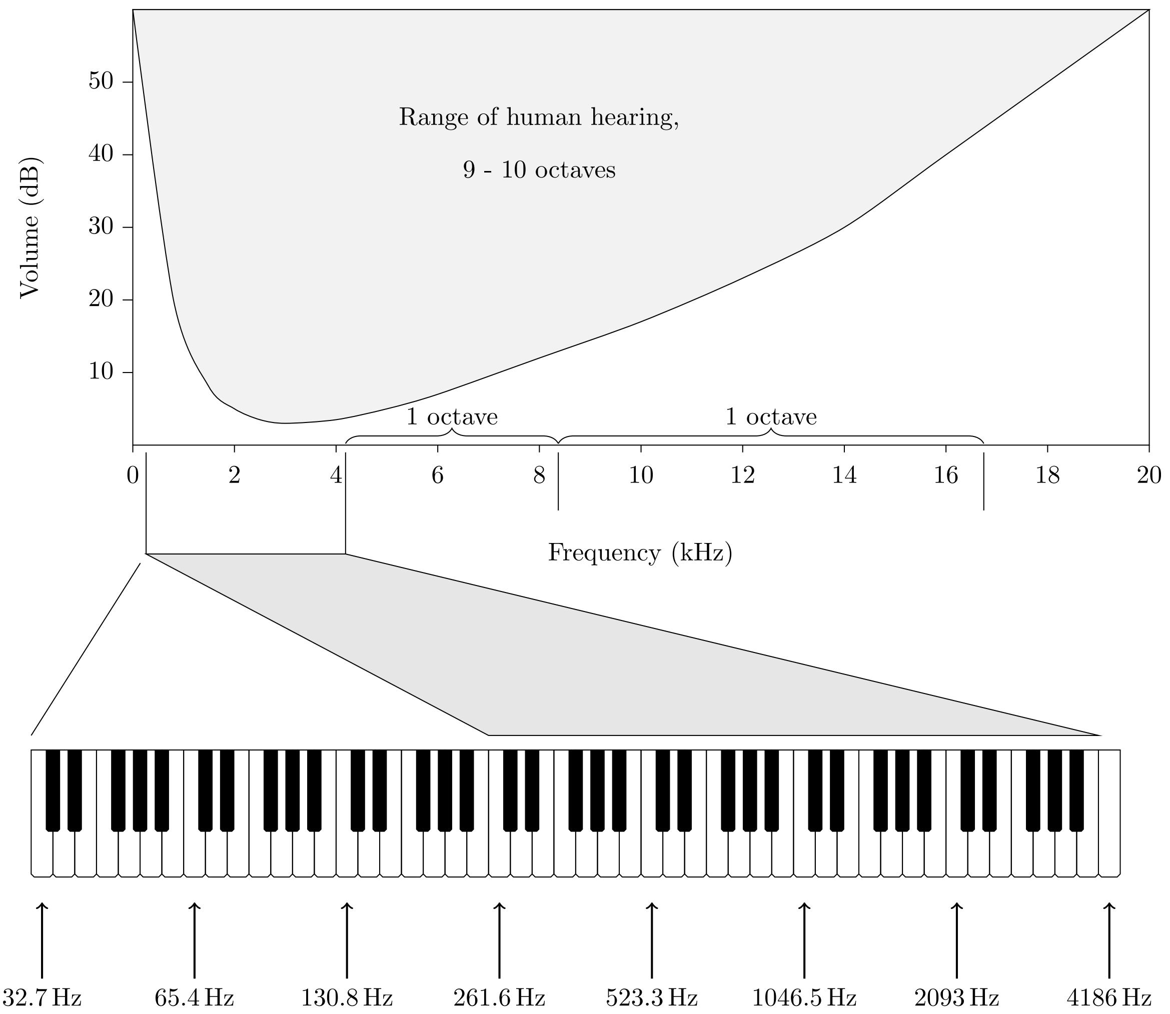
The didactic objective here is that you develop a sense for logarithmic / exponential relationships.

1. Pitch: Perception (octaves) vs physics (frequency); in essence the following relationship (this proviso of “in essence” applies to all the following examples, as in every case there are too many details):



www.youtube.com/watch?v=8BqnN72OlqA (Donald Duck in Mathmagic Land; music starting at 2:50)
www.tinnitracks.com/de/tinnitus/frequenz

Logarithmic perception enables the coverage of an enormous range. Makes sense for young healthy humans: 40-20,000 Hz - Survival of the fittest: the tones in our evolutionary environment were in this range (e.g. whales hear higher frequencies in the sea - for us on land this didn't make sense).



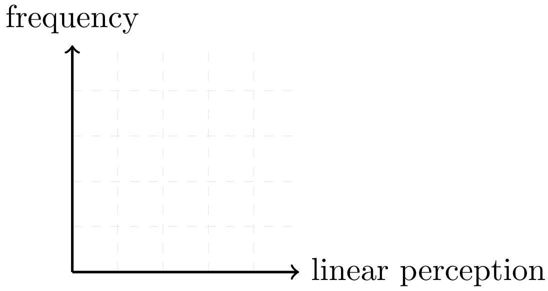
Queen of the Night aria; high note at 1.04: www.youtube.com/watch?v=-quQHNriV-Q

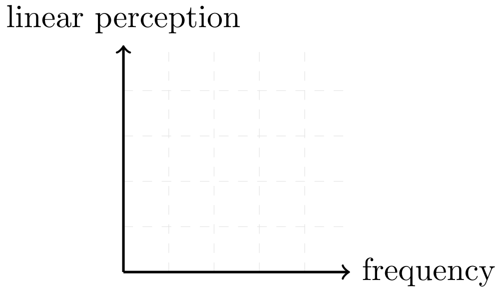
https://en.wikipedia.org/wiki/Florence_Foster_Jenkins_(film) very good

https://en.wikipedia.org/wiki/Marguerite_(2015_film) even better

https://schweizermonat.ch/mozarts-hoehepunkt/

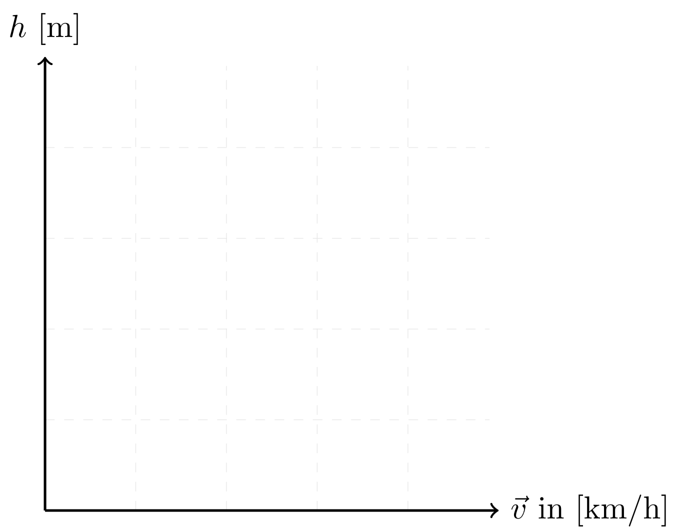
2. On the colloquial use of *logarithmic* and *exponential* as synonyms





3. Richter scale for earthquakes; essentially the following relationship:

4. The barometric formula and the crash of Air France 447; essentially the following relationship:



A column of mine about this (in German) in “Schweizer Monat”:

https://schweizermonat.ch/10-000-meter-900-km-h/

and further articles:

https://en.wikipedia.org/wiki/Barometric_formula

http://en.wikipedia.org/wiki/Air-France-Flight_447

5. pH value:

https://schweizermonat.ch/ph-wert-was-die-zahl-auf-ihrer-hautcremetube-bedeutet/

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

Take care with $f(x-a)$ and $f(cx)$: counterintuitive!

"*semi*-logarithmic" is the same thing as "*simply* logarithmic" (historical error in the naming convention); contrast with *doubly* logarithmic.

Why log scales:
1. In the natural sciences we often have measurements which by their nature are greater than 0 (weights, lengths) and which also span an enormous range (see frequencies, next page). The logarithm is defined for numbers > 0 and a logarithmic scale lets us represent the whole range quite compactly (the physics stays the same - it's only a question of graphical representation).
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