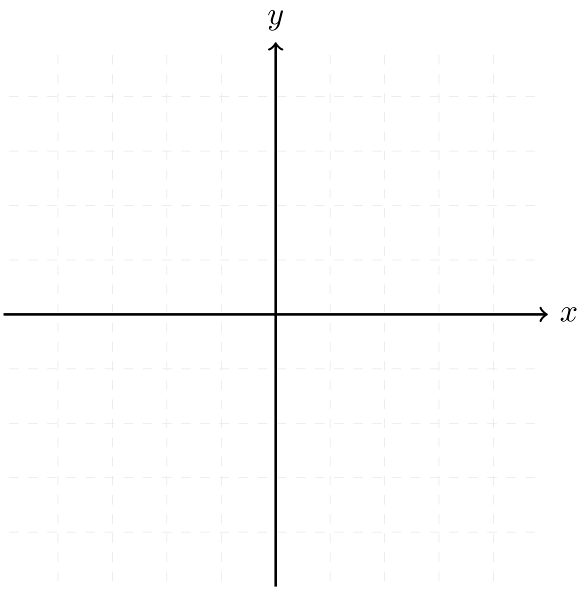
# 16. SELECTED METHODS FOR SOLVING

(16.2) Initial conditions (Anfangsbedingungen)

$y' = 2x$ ("$y$ fehlt")

$y' = y$

$y' = (2x)y$



## (16.3) Slope field (Richtungsfeld)

Again we consider the general differential equation

$$y' = F(x, y) \; . \qquad \text{[e.g. } y' = (2x)y\text{]}$$

and in addition, we let $y = f(x)$ be a solution of this equation [$y = e^{x^2}$, (Konstante $K = 1$)].

STEP 1: We want to investigate the graph of $f$. To this end, we consider the point $(x_0, y_0)$ with $y_0 = f(x_0)$, which of course lies on the graph of $f$ [e.g. $(1, e)$].

STEP 2: But $f$ is a solution of the above differential equation. It follows that

$$f'(x_0) = F(x_0, y_0) \; . \qquad \text{[e.g. } (2x_0)y_0 = (2 \cdot 1)e = 2e\text{]}$$

STEP 3: The number $f'(x_0)$ is then exactly the slope of the graph of $f$ (more precisely: of the tangent to the graph of $f$) at the point $(x_0, y_0)$. In other words: the solution curve which passes through $(x_0, y_0)$ has at this point the slope $F(x_0, y_0)$.

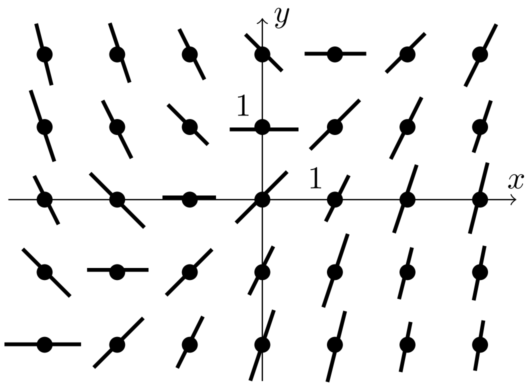
STEP 4: The crucial point here is that we can compute this slope even without knowing the solution function $f(x)$, since of course $F(x, y)$ is given!

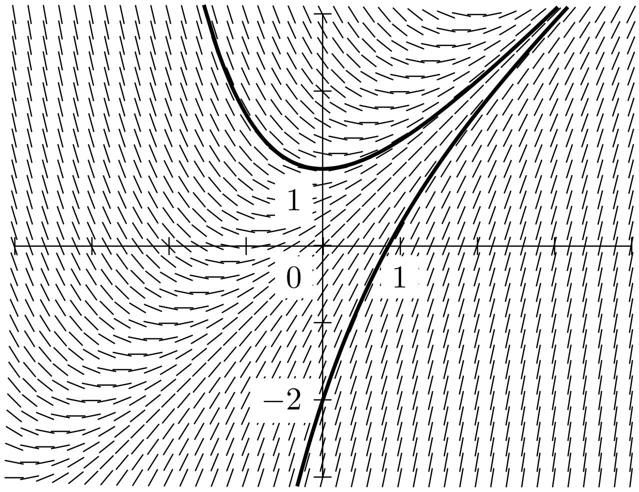
Example

Let $y' = x - y + 1$. (In (16.6.1) we will solve this equation explicitly). First we tabulate $y'$ as it depends on $x$ and $y$.

| $y$ \ $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $2$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
| $1$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
| $0$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $-1$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
| $-2$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |

Now through every point $(x, y)$ of the plane we draw a short line segment with the corresponding slope $y' = F(x, y) = x - y + 1$. At the point $(1, 1)$ the line segment has the slope $F(1, 1) = 1$, at $(0, 2)$ the slope $F(0, 2) = -1$, and so on. Doing this by hand, we obtain the picture on the left. Of course the calculation and drawing can also be delegated to a computer, producing more comprehensive graphs such as on the right.




⊠

The object represented by both the above sketches is called the *slope field* (or *direction field*; German: Richtungsfeld) of the differential equation. If (as in the second case) sufficiently many of these "direction elements" are drawn in, it becomes possible to identify the approximate shapes of solution curves. The two curves which have been added are examples of particular solutions of the differential equation. Since the upper curve passes through the point $(0, 1)$ (among others), this is the particular solution corresponding to the initial condition $(x_0, y_0) = (0, 1)$; similarly, the lower curve corresponds e.g. to the initial condition $(x_0, y_0) = (0, -2)$. More generally, we see that every point in the plane has a solution curve passing through it.

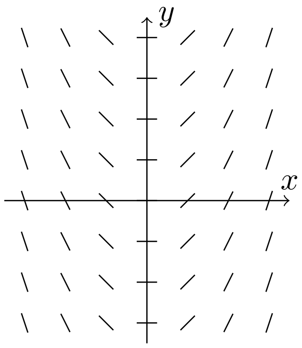
The development of this solution curve can be informally understood as follows: We begin at $(x_0, y_0)$, move in the direction $(1, y')$ ($y' = F(x_0, y_0)$) just a short distance to a new point $(x_1, y_1)$; there we compute the new slope $F(x_1, y_1)$, proceed in the direction $(1, y')$ (with our new $y' = F(x_1, y_1)$) to the point $(x_2, y_2)$ and so on. In this way we obtain a chain of line segments; the smaller the steps are, the more closely this approximates the solution curve through $(x_0, y_0)$. In the limit ("infinitely small" steps, handle with caution) the desired solution curve emerges (cf. (21.4)). The procedure outlined here belongs properly to the mathematical field of "numerics"; we will have a little to say about the numerical solution of differential equations in (21.4). For practical purposes this field is of central importance, as differential equations in practice can seldom be solved explicitly (as in Chapter 16) – often we must content ourselves with numerical procedures.

Finally, another look at the slope field leads us to suspect that the function $y = x$ might be a particular solution of the differential equation $y' = x - y + 1$. This can be immediately verified analytically: the fact is that $y' = 1$ and $x - y + 1 = x - x + 1 = 1$, and so for all $x$ the equation $y' = x - y + 1$ is satisfied.

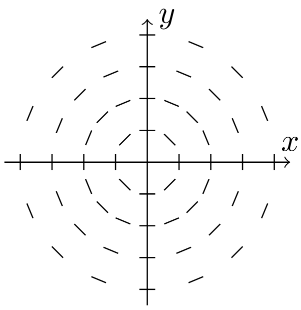
Two more examples follow. In these simple cases the solutions of the differential equation can be guessed by looking at the slope field:

In the first example the solution curves are parabolas (the general solution is in fact $y = \frac{1}{2}x^2 + C$).

In the second example the solution curves are circles centred on the origin (cf. the analytical solution in (16.10.3)).




$y' = x$




$y' = -\dfrac{x}{y}$

A good way to think about DE: from the beginning we receive *local* instructions (the equation), but the *global* picture (slope field, resp. solutions) we do not see at first.

It should be mentioned in closing that these slope fields are not to be confused with the vector fields of (14.3).

## (16.4) First-order linear differential equations

Now we come to the first systematic solving method. This applies to so-called "first-order linear differential equations".

A first-order differential equation is said to be *linear* if it has the form

$$y' = p(x)y + q(x) \ ,$$

where $p$ and $q$ are functions of $x$.

The designation "linear" refers solely to the fact that $y$ (and $y'$) only appear to the power 1. The functions $p(x)$ and $q(x)$, on the contrary, need not be linear at all. "First-order" refers to the fact that only $y$ and $y'$ appear, and not e.g. $y''$.

We may divide linear differential equations into two cases:

A linear differential equation is called *homogeneous* if $q(x) = 0$; otherwise we say it is *inhomogeneous*.

If

$$y' = p(x)y + q(x)$$

is a linear differential equation, then

$$y' = p(x)y$$

is called the *associated homogeneous equation*. For example, $y' = xy + 1$ is an inhomogeneous linear differential equation (of the 1st order) and $y' = xy$ is the associated homogeneous equation.

## (16.5) Solution method for linear differential equations of the 1st order

a) Overview

To solve a linear differential equation, first the associated homogeneous equation is solved and then afterward the "method of variation of constants" is employed to find the solutions of the original equation. We will go through this now in detail.

b) Solution of the homogeneous equation

Let

$$y' = p(x)y$$

be a homogeneous linear differential equation. The solution can be guessed: let $P(x)$ be any antiderivative of $p(x)$ (i.e. $P'(x) = p(x)$). Then

$$\boxed{y = Ke^{P(x)}, \quad K \text{ an arbitrary constant}}$$

is the general solution of the homogeneous equation, since differentiation by applying the chain rule yields

$$y' = Ke^{P(x)} \cdot P'(x) = Ke^{P(x)} \cdot p(x) = p(x)y\ .$$

It should also be noted that the constant function $y = 0$ is always a solution of any homogeneous linear differential equation; set $K = 0$ and check by calculating:

Why not $P(x) + C$, i.e. $Ke^{P(x)+C}$?

<u>Examples</u>

1. $y' = y \sin x$ : $p(x) = \sin x,\ P(x) = -\cos x$; solution: $y = Ke^{-\cos x}$; check by calculating:

2. $y' = e^x y,\ x > 0$.

c) <u>Solution of the inhomogeneous equation ("variation of constants")</u>

We consider once again the inhomogeneous linear differential equation

$$y' = p(x)y + q(x) \tag{1}$$

and the associated homogeneous equation

$$y' = p(x)y \ . \tag{2}$$

As just discussed, (2) has the general solution

$$y = Ke^{P(x)} \ , \tag{3}$$

where $P(x)$ is an antiderivative of $p(x)$.

Now we apply the method of *"variation of constants"* . This method consists of *assuming* that the solution of the inhomogeneous equation (1) has the form

$$\boxed{y = K(x)e^{P(x)}} \ . \tag{4}$$

In other words, we replace the constant $K$ in (3) with some (for the moment still unknown) function $K(x)$, and then try to determine this function in such a way that (4) is a solution of (1).

Of course in mathematics one cannot simply do such things. But just as with Shakespeare's Hamlet: "Though this be madness, yet there is method in 't." – and indeed: it works! You can happily check it at the end yourself by substitution.

To find this function $K(x)$, we take the derivative of $y$ (product rule and chain rule!):

$$y' = K'(x)e^{P(x)} + K(x)p(x)e^{P(x)} \ . \tag{5}$$

Here we have also used the fact that $P'(x) = p(x)$. Now we substitute $y$ from (4) and $y'$ from (5) into the original equation

$$y' = p(x)y + q(x) \ .$$

We obtain

$$K'(x)e^{P(x)} + \underline{K(x)p(x)e^{P(x)}} = \underline{p(x)K(x)e^{P(x)}} + q(x) \ .$$

The underlined terms add to zero (which was the point of this approach) [a checkpoint for the exam!] and all that remains is

$$K'(x)e^{P(x)} = q(x) \ .$$

By multiplying with the reciprocal $e^{-P(x)}$ of $e^{P(x)}$ we obtain

$$K'(x) = q(x)e^{-P(x)} ,$$

from which we can compute $K(x)$ as an antiderivative of $q(x)e^{-P(x)}$:

$$K(x) = \int q(x)e^{-P(x)}\,dx + C .$$

[Add $+C$ exactly here!] Finally, we substitute the expression thus obtained for $K(x)$ into the initial assumption (4) and find as a solution of the inhomogeneous linear differential equation

$$y = \left(\int q(x)e^{-P(x)}\,dx + C\right) e^{P(x)} ,$$

[not here at the end!] where $P(x)$ is an arbitrary antiderivative of $p(x)$. Formulated differently:

(6) $$y = (K_0(x) + C)e^{P(x)} ,$$

where $K_0(x)$ is an arbitrary, fixed antiderivative of $q(x)e^{-P(x)}$.

The method for solving an inhomogeneous linear differential equation

$$y' = p(x)y + q(x)$$

can thus be summarised as follows:

1) Solve the associated homogeneous differential equation

$$y' = p(x)y .$$

The solution has the form

$$y = Ke^{P(x)} ,$$

where $P(x)$ is an antiderivative of $p(x)$.

2) Vary the constant $K$, i.e. make the assumption

$$y = K(x)e^{P(x)} ,$$

where the function $K(x)$ still has to be determined.

3) If you apply this assumption to the original inhomogeneous differential equation, after some rearrangement you get an expression for $K'(x)$; from this, try to compute $K(x)$.

$\boxed{\text{(16.6) Variation of constants: examples (different examples than in the textbook)}}$

1. $y' = -y + e^x$, in general: $y' = p(x)y + q(x)$

2. $y' = \frac{y}{x} + 1 \quad (x > 0)$, in general: $y' = p(x)y + q(x)$

Aufgabe 16-3d); to solve yourself: $y' = \frac{3}{x}y + x, x > 0$, in general: $y' = p(x)y + q(x)$

## (16.9) Separation of Variables

We proceed now to another systematic solving method, the method of *separation of variables*. This method can always be applied if the differential equation has the form

$$\boxed{y' = r(x)s(y)}$$

i.e. when the right side $F(x, y)$ can be written as the product of two functions of one variable (one function of $x$ and one function of $y$). As a special case, the functions $r(x)$ resp. $s(y)$ may also have the constant value 1, which means that the differential equations

$$y' = r(x) \quad \text{and} \quad y' = s(y)$$

are also subject to this method.

We illustrate the method by means of a simple, familiar example; $y' = \alpha y$, where $y > 0$ and $t > 0$; $\alpha \in \mathbb{R}$:

Note 1) $y' = \alpha y$ is a sensible model for many things; but this leads apparently (see above) *necessarily* to exponential growth!
Note 2) Through $y' = \alpha y$ we find we automatically have to deal with ln and later "$e$ to the power" (see above). ln and $e$ are therefore necessary; not possible to do everything with $\log_2$ and $\log_{10}$ resp. $2^n$ and $10^n$.

In this situation we proceed according to the following method:

1) Write the differential equation in the form

$$\frac{dy}{dx} = r(x)s(y) \ .$$

2) Bring all terms with $y$ onto the left side, all terms with $x$ to the right side. In the process, we should formally multiply by $dx$:

$$\frac{dy}{s(y)} = r(x)\,dx \ .$$

3) Form the indefinite integral of both sides:

$$\int \frac{dy}{s(y)} = \int r(x)\,dx + C \ .$$

Do not forget the constant of integration $C$!*

This gives you the equation

$$S(y) = R(x) + C \ ,$$

where $S(y)$ is an antiderivative of $\dfrac{1}{s(y)}$, $R(x)$ an antiderivative of $r(x)$. Through this equation, $y$ is given "implicitly" as a function of $x$. This $y$ is then the general solution of the differential equation.

4) Solve for $y$ (if possible).
5) Check whether there are constant solutions which were not already included.

Comment on 5); find constant solutions of $y' = x(y^2 - 4)$.

* Properly speaking, each integral gives rise to such a constant; we combine them into one though.

## (16.10) Examples of separation of variables

1. The linear homogeneous differential equation

We have already seen in (16.5.b) that the differential equation

$$y' = p(x)y$$

has the general solution

$$y = Ke^{P(x)}$$

where $P(x)$ is an antiderivative of $p(x)$.

This equation can also be solved using separation of variables. We demonstrate this possibility for a concrete example, namely for the equation

$$y' = x^2 y \ ,$$

and to do so we follow the five steps of the "recipe" in (16.9) one at a time.

2. The linear differential equation with constant coefficients

The differential equation

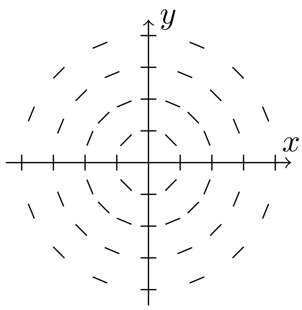
$$y' = ay + b,\ a, b \in \mathbb{R}, \quad a \neq 0$$

was already solved in the textbook (not the lecture) in (16.8). Our new method proves effective here as well though. We write

$$y' = a(y + \frac{b}{a})$$

(i.e. we take $r(x) = a,\ s(y) = y + \frac{b}{a}$).

3. $y' = -\dfrac{x}{y}$ where $y \neq 0$



4. $y' = e^{x-y}$ (not in the textbook)

Terminology: in *this* course we use the following: a **particular** solution (German: spezielle Lösung) is a solution which satisfies given initial conditions (in contrast to the **general** solution (German: allgemeine Lösung)). **Constant (=stationary)** solutions are solutions which do not vary over time. **Singular** solutions are those which are not captured as part of the general solution using the method of separation of variables - these are also constant solutions (example in the textbook: (16.10.4)).

**You can always check proposed solutions:**

**if you think you have found a solution of a differential equation:**

**substitute it into the DE to check!**

**Important:**

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

**Lessons learnt:**

A first-order differential equation is said to be *linear* if it has the form

$$y' = p(x)y + q(x) \; ,$$

where $p$ and $q$ are functions of $x$.

2 procedures: Variation of constants in (16.5) and separation of variables in (16.9).

Terminology: in *this* course we use the following: a **particular** solution (German: spezielle Lösung) is a solution which satisfies given initial conditions (in contrast to the **general** solution (German: allgemeine Lösung)). **Constant (=stationary)** solutions are solutions which do not vary over time. **Singular** solutions are those which are not captured as part of the general solution using the method of separation of variables - these are also constant solutions (example in the textbook: (16.10.4)).

**You can always check proposed solutions:**

**if you think you have found a solution of a differential equation:**

**substitute it into the DE to check!**