D. DIFFERENTIAL EQUATIONS (DE)

15. THE CONCEPT OF A DIFFERENTIAL EQUATION (DIFFERENTIALGLEICHUNG)

For many students the material up to this point will have been repetition (with the exceptions of chapters 8 and 14). Differential equations (DEs), our next topic, are new ground. DEs are centrally important as an instrument for modelling in the natural sciences, in engineering and in economics. DEs are deterministic approaches to modelling; the stochastic (random) approaches will follow in the spring semester.

New and confusing about DEs is that we have everything in a single equation: the derivative y' (which just as before *also* represents the slope of the graph), the function y itself, and the argument x, for example in the form

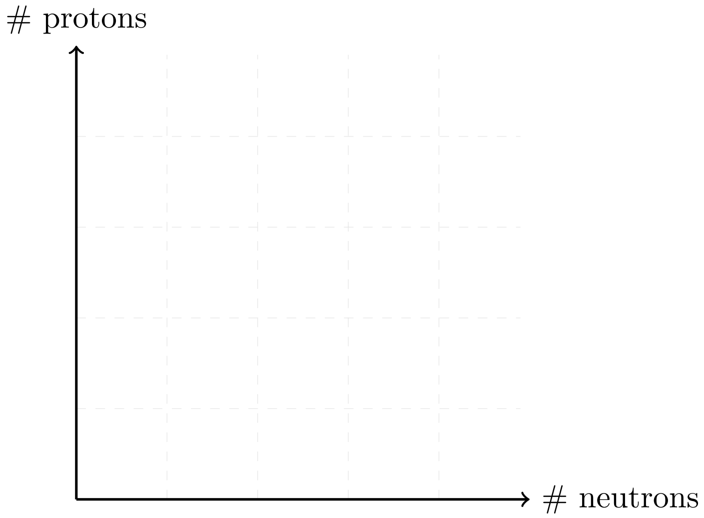
$$y' = \frac{y}{x}.$$

Another jarring novelty is that the solution is a function $y(x)$ and not as elsewhere a number. Also central to the applied sciences is the ability to frame a DE as such. We will practice this in Chapter 15 using some applications.

(15.0) General information about radioactive decay – not in the book

Periodic table with excess neutrons at high atomic numbers:

$$\frac{N}{Z} \doteq 1 + 0.015(N+Z)^{2/3}$$



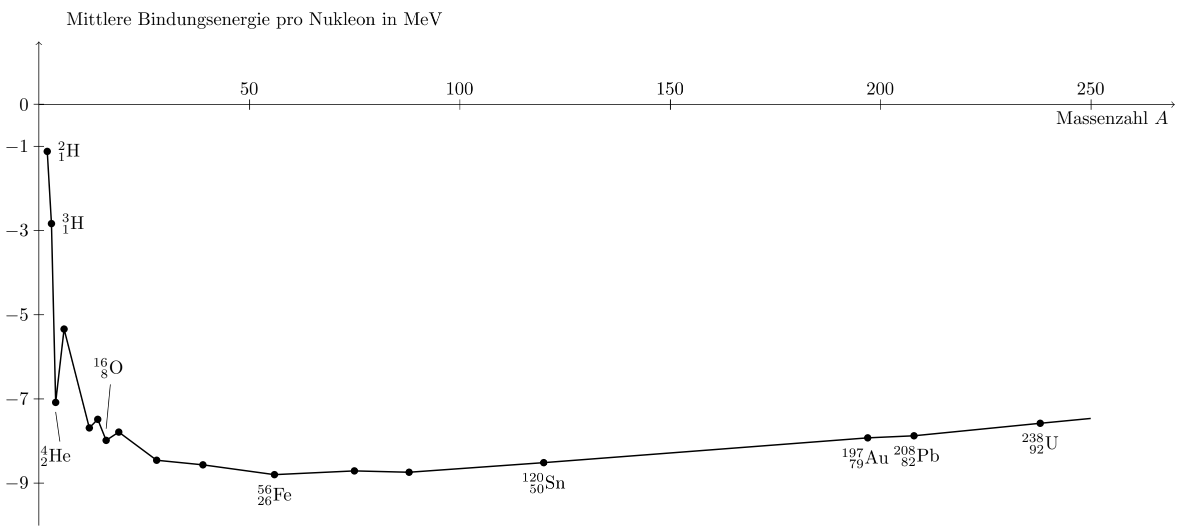
If they are too far outside the valley of stability, isotopes decay through one of the following: α decay (from $N+Z>200$), β^- decay (when there are too many neutrons), β^+ decay (when there are too many protons); γ decay then accompanies the above. The abovementioned valley of stability exists.

To motivate the first DE which we will examine (15.3 a), we look now at the *instant of decay of an isotope.* The time until the decay of a given individual isotope is random, unpredictable, and memoryless. One speaks of spontaneous decay (spontaner Zerfall), occurring until a stable condition is reached. The atoms decay independently of each other (as long as no chain reactions occur). On the basis of these physical findings we conclude that, of one kilogramm of Pu239, a certain quantity will decay in the next hour. If we wait 2 hours, about twice as much will decay. If in addition we double (or halve) the initial amount present, then twice (half) as much will decay. **Summary: the decrease in quantity of Pu239 is proportional to the elapsed time and to the current quantity.** In the coming pages we will see how this result informs our modelling.

A short digression on nuclear energy / nuclear weapons: the 2 central reasons why we have nuclear energy and nuclear weapons (first half of the 20th century; by the time of WW2 it became clear that one can build a bomb from this; historic bad news for Japan):

* If heavy isotopes are split, we have excess neutrons. These excess neutrons can then split further atoms, resulting in a chain reaction.

* The energy is gained according to $E = mc^2$ (Albert Einstein) from the loss of mass (the split pieces are lighter than the original! - the rest was converted into energy; such a thing by definition does not occur in chemistry (law of conservation of mass)). In this context we have the following curve:



https://schweizermonat.ch/endlich-unsere-energieprobleme-sind-geloest/

(15.3) Examples of differential equations

a) Radioactive decay

To recap quickly, we need:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or rather, with time as the variable:

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

a1) Modelling

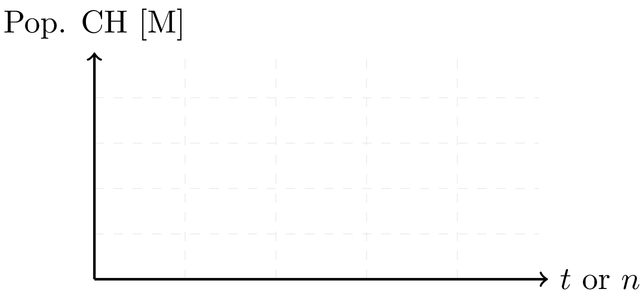
a2) Observations

What is the significance of the K in $y(t) = Ke^{-\lambda t}$? We take $t = 0$ and obtain:

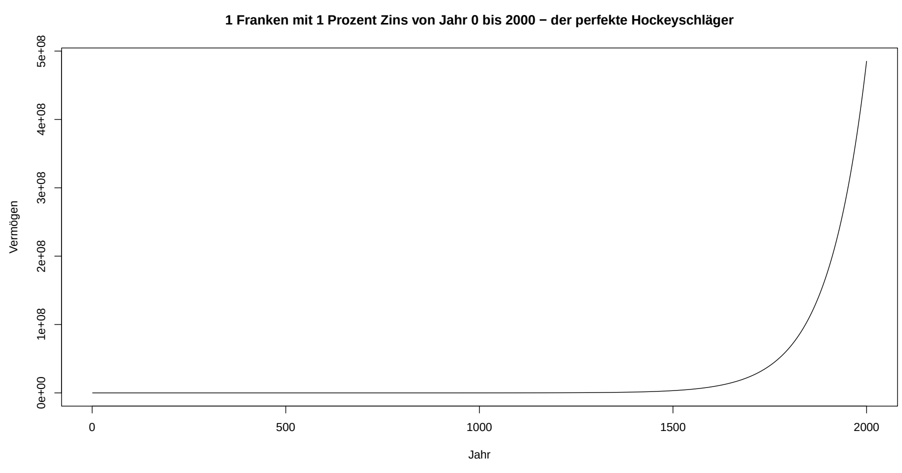
Terminology:

* $N(0) = N_0$ is called an **i**nitial **c**ondition, IC (in German: **A**nfangs**b**edingung, AB)
* $N(t) = Ke^{-\lambda t}$ is called the general solution
* $N(t) = N_0 e^{-\lambda t}$ is called the particular solution with the IC $N(0) = N_0$.

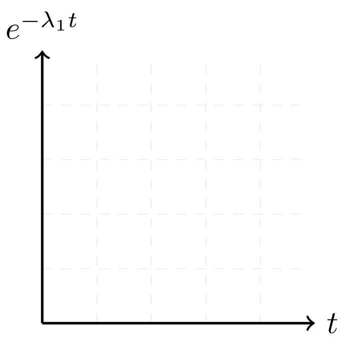
A few notes on exponential resp. geometric behaviour (next 6 pages); see also www.schweizermonat.ch/alles-waechst-exponentiell.

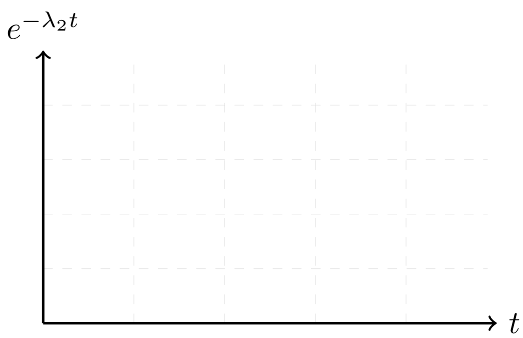


	increasing	decreasing
exp (continuous)		
geom (discrete)		

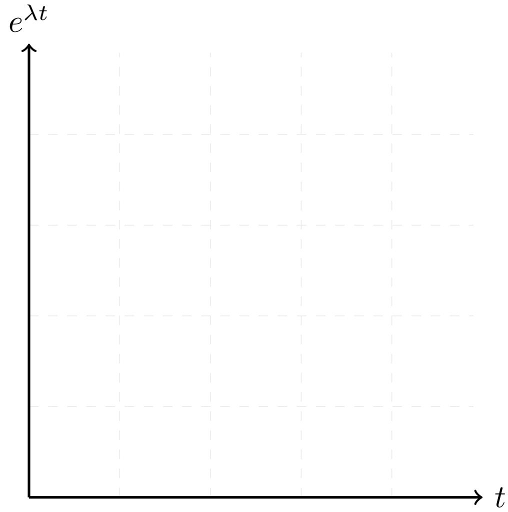


Half-life period $T_{1/2}$ (German: Halbwertszeit)





And now on the increase: Doubling period T_2



After one year of a mathematics course, one can prove the following characteristics of geom/exp behaviour:

Constant time to double **if and only if** we have exponential (continuous) or geometric (discrete) growth.

and analogously

Constant time to halve **if and only if** we have exponential (continuous) or geometric (discrete) decrease.

Furthermore, the following assertion holds for geometric or exponential behaviour: adding/subtracting along the x-axis is the same as multiplying/dividing along the y-axis.

Radioactive waste: misconception and truth

downright foolish: “after 30,000 years all the waste has decayed”: first of all that should be 24,110 years (half-life of Pu239) and anyway: that’s only the *half*-life $->$ the (other) half of it will still be there!

slightly more intelligent: “in a constant span of time the remaining quantity is halved”: this is true if we only have/consider the isotope Pu239 and if no new Pu239 is being produced. Then one could argue that the quantity quickly goes to 0, because $e^{-\lambda t}$ goes very quickly to 0 (this last is true - but it would still be a very long time):

Aside: when we say that something “goes quickly to 0”, we typically mean that in comparison to a power relation such as

$$\frac{1}{t^\alpha}$$

where $\alpha > 0$. Powers (in the denominator) do in fact go to 0 more slowly in the long term!

On the next page you can see how the highly radioactive waste (Hochaktiver Abfall, HAA) from the annual production of 1 GW (electrical) decays. Note the following 5 points with regard to this graph:

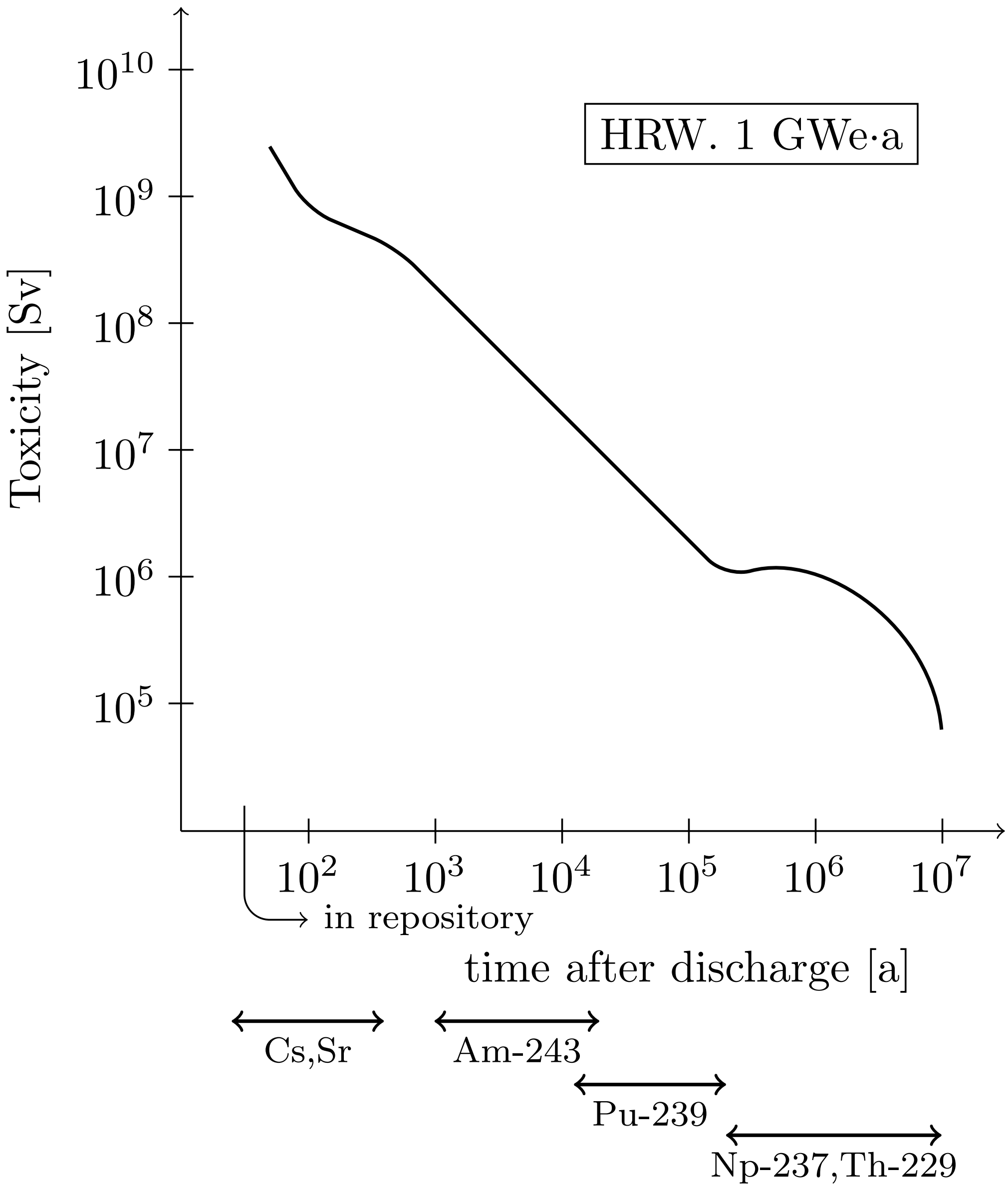
1. Radioactive materials are being created over and over again as products of decay chains.

2. From the start, a mixture of various radioactive isotopes is present.

3. This is graphed on log-log axes. Since the result is more or less a straight line, it follows (see Chapter 18) that this is in reality a power relation - so a slow decline.

4. Along the y-axis we do not have the remaining isotopes themselves, but rather the toxicity in sieverts. This takes into account their harmfulness to humans; the energy of the radiation, whether α, β, γ-radiation and from which isotopes (this also helps us know into which organs the isotopes would be absorbed if ingested).

5. Instead of *one* repository site, if we divided the same material among 100 sites (there are many reasons not to do this), the toxicity would also decrease by a factor of 100 (along a log-10 axis by 2 units).



HRW. 1 GWe·a
Toxicity [Sv]
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{2} 10^{3} 10^{4} 10^{5} 10^{6} 10^{7}
in repository
time after discharge [a]
Cs,Sr
Am-243
Pu-239
Np-237,Th-229

The C-14 dating method (Nobel prize: Willard Libby)

C12, C14 ($T_{1/2} = 5730$ years, extremely rare)

$$y' = -\lambda y \quad \text{and} \quad \text{therefore} \quad y(t) = Ke^{-\lambda t}$$

Plants, animals (organic origin, "C")

C12 and C14 are *biologically* identical

Plants and animals incorporate C12 and C14;
once dead: their C14 decays, while in the atmosphere
it is constantly being renewed through cosmic
radiation and spallation of N14.

When remains are found: measure the ratio
of C14 to C12, and find t (see book)

for 300 - 60'000 years this is usable, beyond that
we are below the measurable threshold

Problems of this method - please look up details in technical literature:

1. False results due to solar activity

2. False results due to remaining radionuclides from *surface* tests of atomic weapons before 1963: as a result of the atom bomb tests before 1963, the concentration of C14 relative to C12 was nearly doubled:
https://upload.wikimedia.org/wikipedia/commons/e/e2/Radiocarbon_bomb_spike.svg

3. Dating the wood in an old house tells us when the tree was cut down, not when the house was built....

4. Combustion of *fossil fuels* increases the proportion of C14-free CO2 (fossil fuel is old, and thus almost solely C12).

5. https://schweizermonat.ch/wie-wissen-wir-dass-oetzi-5300-jahre-alt-ist

And with that we'll stop listing corollaries to $e^{\lambda t}$.

Now we will look at many more examples in which DEs make themselves useful as suitable modelling tools. In Chapter 15 we will concern ourselves mainly with the deployment of DEs. Their solution - if at all - will follow in chapter 16.

b) Population growth

1) $N(t)$ = number of individuals at time t:
Lynx in canton Glarus
Current assumption: the *absolute* increase
is proportional to the current population, i.e.:

2) unrestricted growth is (frequently) unrealistic
A fixed upper limit $B, B > 0$ is often a sensible model
"carrying capacity" of the habitat
$0 \leq N(t) \leq B$ and in fact:

3) Combining the two models above:

Solution in the book: sigmoid curves (a special case is the "logistic function")

c) Spread of an infectious disease (an initial, trivial model; see next page)

G healthy, N infected; $G + N = B$
no recovery, not fatal, so B is constant
here: infected = infectious = sick (and in fact immediately so)

Two didactic tips:

* start with those infected and
* "infection = success"

$N'(t)$ is proportional to the number of *sufficiently close* contacts between healthy G and infected N

AsideB: "sufficiently close" (contact) is a practical, customary formulation in mathematical epidemiology. It simply means, that the contact is enough to result in an infection (e.g. HIV: body fluids, Covid19: coughs and sneezes, mediaeval superstition: eye contact.)

Then $N'(t)$ is proportional to the *number* $N(t)$ of infected people and to the *proportion* of healthy people

$$\frac{B - N(t))}{B} \, ,$$

thus

e) Bimolecular reactions (molecule-molecule reactions)

a, b concentration of A, B initially; $x(t)$ the concentration of C at time t.

Law of mass action (special case with 1 reaction product, 1st-semester chemistry):

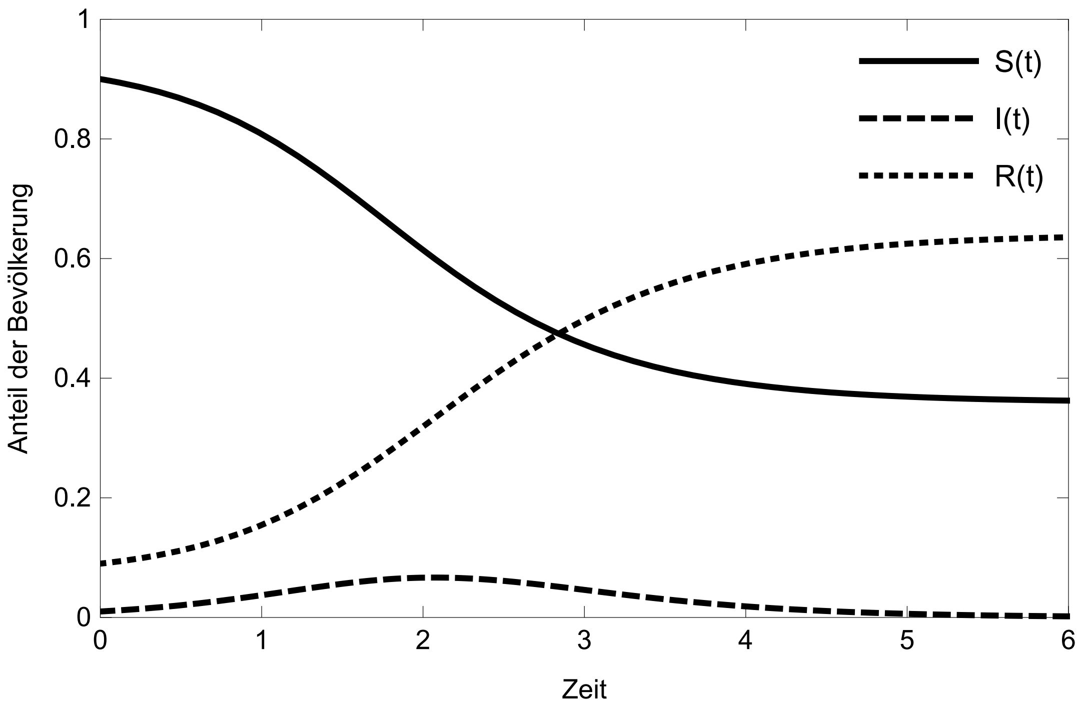
Examples of *systems* of DEs:

Can you guess a solution of:

SIR (a standard model - not directly applicable to Covid-19, but numerically solvable)

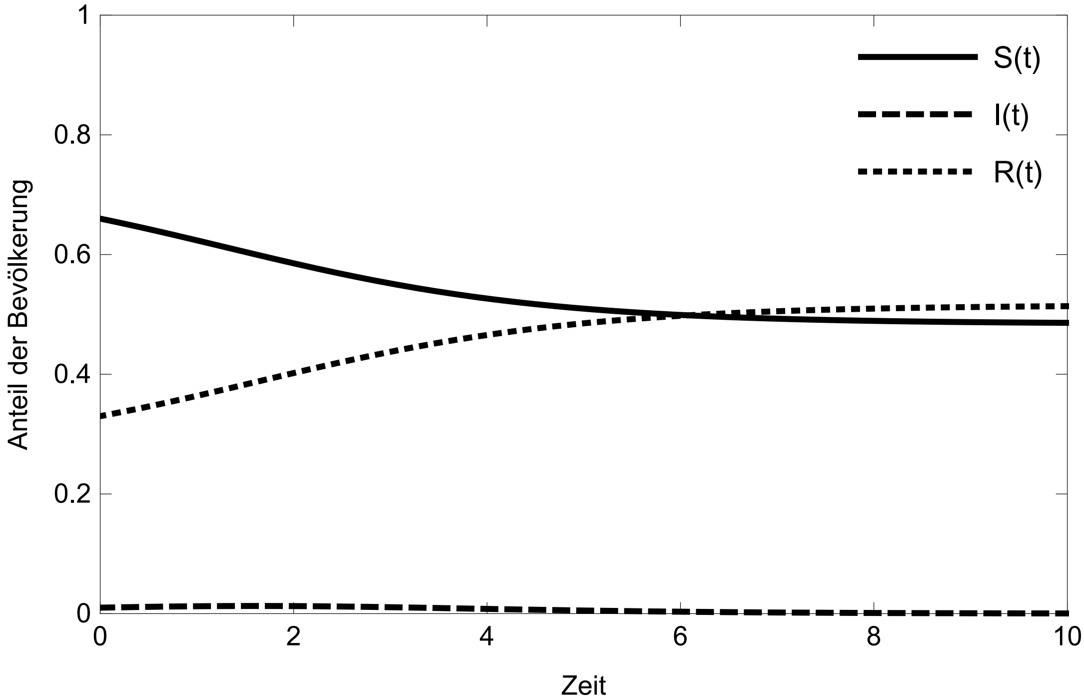
<u>the academic world:</u> $R_0 := \frac{\lambda}{\mu}$; so-called Basic Reproduction Ratio: number of new infections due to a single infected person under optimal conditions; i.e. $S(0) \doteq 1$.

Politics:



H.I.T: Herd Immunity Threshold:

e.g. extreme R_0 (see also Wikipedia):



This topic (and many others from Chapter 15) can absolutely be treated at secondary school ("Gymnasium") level, for example in the context of a project week, special course or "Maturaarbeit" with mathematics or physics teachers.

A few tips on scientific communication:

* relevant to the Science Faculty: climate, epidemics, nuclear power stations (operation, waste)

* reality vs model (SIR) vs simulation (see the 2 graphs above)
* model risk: $S > I > R$ or $S > I > R$ and then again to S?
* does the target audience (politicians, journalists) understand the model assumptions
* to reduce complexity: use formulations like "the main effect is"; "in this lecture/article assume that.... (unless we explicitly discuss otherwise)" - but it always has to be true: you as the scientist must have the last word!
* www.reach.ch

www.luchsinger-mathematics.ch/as10.pdf
www.luchsinger-mathematics.ch/me.html
https://schweizermonat.ch/wie-man-eine-epidemie-stoppt
https://de.wikipedia.org/wiki/Schwarzer_Tod (zeigen)
Lecture of Christoph Luchsinger for Science Alumni UZH, 15. April 2021:
https://www.sciencealumni.uzh.ch/de/Veranstaltungsarchiv.html

Lanchester DE: Once Upon a Time in the West vs High Noon - parallel or serial?

More at `https://schweizermonat.ch/wo-ist-frank/` .

Lotka-Volterra equations (also called predator-prey equations, voluntary): see the web or as mentioned, Dr Luchsinger's old Basel lectures

What could be asked on the exam about *systems* of DEs?

(15.4) General information about differential equations

→ read in the book

What kinds of DEs are possible? A few tips:

* ordinary and partial (we won't deal with the latter)

* 1st order with y', y and 2nd order with y'', y', y (again, we won't deal with the latter - except to verify existing solutions or search for constant solutions)

The general form of an explicit first-order DE is $y' = F(x, y)$, where F is a function of two variables. We will see plenty of examples, such as for instance

$$
\begin{aligned}
&y' = \frac{2y}{x} \\
&y' = x - y + 1, \quad \text{etc. .}
\end{aligned}
$$

Before going further we state formally what we mean by a *solution* of the DE $y' = F(x, y)$: a function $y = f(x)$ is called a solution of this DE if, for all x in the domain of definition D, it holds that $f'(x) = F(x, f(x))$.

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

A solution of a DE is a function $y(x)$ (or $y(t)$) and not as elsewhere a number!

Terminology (using the example of exponential growth):
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rate vs. half-life / doubling time:

$$\lambda T_{1/2} = \ln(2) \qquad \lambda T_2 = \ln(2)$$

What kinds of DEs are possible? A few tips:

* ordinary and partial (we won't deal with the latter)

* 1st order with y', y and 2nd order with y'', y', y (again, we won't deal with the latter - except to verify existing solutions or search for constant solutions)

The general form of an explicit first-order DE is $y' = F(x, y)$, where F is a function of two variables. We will see plenty of examples, such as for instance

$$\begin{aligned} y' &= \frac{2y}{x} \\ y' &= x - y + 1, \quad \text{etc. .} \end{aligned}$$

Before going further we state formally what we mean by a *solution* of the DE $y' = F(x, y)$: a function $y = f(x)$ is called a solution of this DE if, for all x in the domain of definition D, it holds that $f'(x) = F(x, f(x))$.