

This script is an extract, with gaps, from the book “Introduction to the Mathematical Treatment of the Natural Sciences I - Analysis” by Christoph Luchsinger and Hans Heiner Storrer, Birkhäuser Scripts. As a student you should buy the book as well and work your way through it completely during the course MAT 182. You are allowed to save this PDF and modify it as you like, for your own use during MAT 182. For further use outside of MAT 182, please contact the lecturer, Christoph Luchsinger of the University of Zürich, in advance. The copyright remains with Birkhäuser!

13. FURTHER TECHNIQUES OF INTEGRATION

(13.2) Substitution

This method of integration is obtained by reversing the *chain rule* of differential calculus. With slightly different notation to that in (5.2), this rule states that

$$F(u(x))' = F'(u(x)) \cdot u'(x) .$$

In this formula we now we set $F' = f$. In particular, F becomes an antiderivative of f . Now the formula reads as follows:

$$F(u(x))' = f(u(x)) \cdot u'(x) .$$

From this we see that the composite function

$$x \mapsto F(u(x))$$

is an antiderivative of the (relatively complicated-looking) function

$$x \mapsto f(u(x)) \cdot u'(x)$$

or to put it differently:

$$\int f(u(x))u'(x) dx = F(u(x)) + C ,$$

where F is an arbitrary antiderivative of f .

Despite the elaborateness of its integrand, this formula has many important applications. The process is as follows: Suppose $\int g(x) dx$ is the desired integral. We look for a way to write the integrand $g(x)$ in the form

$$g(x) = f(u(x)) \cdot u'(x)$$

for some suitably chosen functions f and u . Above all we must take care that the function $u(x)$ and its derivative $u'(x)$ both appear in the integrand $g(x)$. A solid mastery of the rules of differentiation is indispensable here! Once we have determined f and u , it only remains to find an antiderivative F of f and apply it to $u(x)$. $F(u(x))$ is then an antiderivative of $g(x)$:

$$\int g(x) dx = \int f(u(x)) u'(x) dx = F(u(x)) + C .$$

This procedure will now be demonstrated with a few examples.

(13.3) Examples of the substitution rule - in line with those in the textbook

1. $I = \int \cos(x^3) \cdot 3x^2 dx .$

2. And what can we do with $J = \int \cos(x^3) \cdot x^2 dx ?$

3. For those who have serious difficulty in finding the right f , u , and u' : write the integrand $g(x)$ explicitly as a product of 2 factors and then think about which of these factors can be an $f(u(x))$, if the other one has to be its $u'(x)$ - or vice versa. We will pay attention to this in the following exercises.

We can even integrate such horrors as $\int \frac{\sin x}{\sqrt{\cos x}} dx$ in this way:

How do the physicists calculate here?

In practice we can summarise the procedure as follows:

- 1) Choose $u(x)$. Set $du = u'(x) dx$.

(This relation is obtained in a completely formal way from

$$\frac{du}{dx} = u'(x)$$

by multiplication by dx .)

- 2) In the desired integral, replace $u(x)$ with u and $u'(x) dx$ with du .
- 3) The integral now has the form

$$\int f(u) du .$$

Determine an antiderivative

$$\int f(u) du = F(u) + C .$$

- 4) In the expression $F(u)$, replace the quantity u once again with $u(x)$. The desired integral is then equal to $F(u(x)) + C$.

We illustrate the process according to points 1) through 4) above with the following examples.

4. $\int \sin x \cos x dx$

6. $\int \frac{1}{x^2} e^{1/x} dx$

(13.4) The substitution rule for definite integrals

Because $F(u(x))$ is an antiderivative of $f(u(x)) \cdot u'(x)$, the Fundamental Theorem of Calculus informs us that

$$\int_a^b f(u(x))u'(x) dx = F(u(b)) - F(u(a)) .$$

And since we also have $F(u) = \int f(u) du$ (since F is an antiderivative of f), the right side can also be construed as

$$\int_{u(a)}^{u(b)} f(u) du \quad \left(= F(u(b)) - F(u(a)) \right) .$$

With this we obtain the formula

$$(*) \quad \int_a^b f(u(x))u'(x) dx = \int_{u(a)}^{u(b)} f(u) du .$$

Note the “transformation” of the limits of integration. We have two possibilities, then, which must not be confused:

- If we consider x as the variable (left side of (*)), then at the end we need to return to $F(u(x))$. The limits of integration are then a and b .
- If we consider u as the variable (right side of (*)), then at the end we retain $F(u)$, but need to substitute the new limits of integration $u(a)$ resp. $u(b)$.

In this course we will work exclusively with the first variant. It’s still a good idea to be aware of the second one, in order to be able to recognise it again in future courses. Critical for both forms is that we recognise the structure of the integrand. It needs to appear in this form: $g(x) = f(u(x)) \cdot u'(x)$ and we have to find the antiderivative F of f . The recommended procedure in this course for *definite* integrals with limits a and b is to separate the two steps clearly: find an antiderivative $F(u(x))$ first and then evaluate it at a and b using the formula

$$\int_a^b f(u(x))u'(x) dx = F(u(b)) - F(u(a)) .$$

We revisit Example 6 above with some limits of integration:

$$\int_1^2 \frac{1}{x^2} e^{1/x} dx =$$

(13.5) Integration by parts (partielle Integration)

For this integration method we begin with the *produkt rule* (5.2):

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) .$$

Clearly every antiderivative of the left side is also an antiderivative of the right side. Equally clearly, $f(x)g(x)$ is an antiderivative of the left side. Thus,

$$f(x)g(x) = \int f'(x)g(x) dx + \int f(x)g'(x) dx$$

or, slightly rearranged:

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx .$$

This is the formula for *integration by parts*.

As the letters f and g are often already in use, we prefer to write u and v instead:

$$\int u'(x)v(x) dx = u(x)v(x) - \int u(x)v'(x) dx .$$

This formula can often be used successfully to compute $\int f(x) dx$ if $f(x)$ can be written as a product of two factors, one of which (namely u') has a simple antiderivative (u), and the other of which (v) has a simple derivative (v'). We illustrate the process with a few examples.

Before turning to the examples, we state the formula for integration by parts as it applies to definite integrals. This is obtained simply by inserting the limits of integration:

$$\int_a^b u'(x)v(x) dx = u(x)v(x) \Big|_a^b - \int_a^b u(x)v'(x) dx .$$

(13.6) Integration by parts: examples

- $\int x e^x dx$

In the stochastics course MAT 183 we will need

$$E[X] = \int_0^{\infty} x \lambda e^{-\lambda x} dx,$$

where $\lambda > 0$ (the expected value of an exponentially distributed random variable):

3. In (12.3) the antiderivative $\int \ln x \, dx$ seemed to appear from nowhere; how in the world could we have guessed that? Let's tackle it again now though:

LAPTE: When we integrate by parts, we need to proceed in such a way that the resulting new integral is easier than the old one. If we classify the functions involved according to LAPTE (Logarithms; Algebraic/Polynomials; Trigonometric; Exponential), then we need to take the derivative of whichever factor is further to the left in LAPTE, and integrate whichever factor is further to the right. For example it's useful to take the derivative of the natural logarithm, because then we obtain $1/x$, which we can combine with polynomials or other powers of x if we have any. Trigonometric and exponential functions, on the other hand, remain equally complicated if we have to integrate them.

(13.7) Table of integrals

Function	Antiderivative
a) <u>Rational functions</u>	
$(ax + b)^n \quad a \neq 0, \quad n \neq -1$	$\frac{(ax + b)^{n+1}}{a(n+1)}$
$(ax + b)^{-1} \quad a \neq 0$	$\frac{1}{a} \ln ax + b $
$\frac{1}{ax^2 + 2bx + c} \quad b^2 > ac$	$\frac{1}{2\sqrt{b^2 - ac}} \ln \left \frac{ax + b - \sqrt{b^2 - ac}}{ax + b + \sqrt{b^2 - ac}} \right $
$\frac{1}{ax^2 + 2bx + c} \quad b^2 < ac$	$\frac{1}{\sqrt{ac - b^2}} \arctan \frac{ax + b}{\sqrt{ac - b^2}}$
$\frac{1}{ax^2 + 2bx + c} \quad b^2 = ac$	$-\frac{1}{ax + b}$
$\frac{ax + b}{cx + d} \quad c \neq 0$	$\frac{ax}{c} - \frac{ad - bc}{c^2} \ln cx + d $
b) <u>Square roots</u>	
$\sqrt{x^2 \pm a^2}$	$\frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln \left x + \sqrt{x^2 \pm a^2} \right $
$\sqrt{a^2 - x^2}$	$\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{ a }$
$\frac{1}{\sqrt{x^2 \pm a^2}}$	$\ln \left x + \sqrt{x^2 \pm a^2} \right $
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin \frac{x}{ a }$
c) <u>Trigonometric functions</u>	
$\tan x$	$-\ln \cos x $
$\cot x$	$\ln \sin x $
$\frac{1}{\sin x}$	$\ln \left \tan \frac{x}{2} \right $
$\frac{1}{\cos x}$	$\ln \left \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right $

Examples for the use of the table

A 13-4a)

A 13-5a)

So the methods for finding an antiderivative are: 1. substitution and integration by parts, 2. consult a table of integrals, 3. ask a mathematician, 4. numerical methods (cf. Chapter 21)

Differentiation always works:

when you think you've found an integral,

check by differentiating!

Important:

1. Next, read the corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

Substitution:

$$\int g(x) dx = \int f(u(x)) u'(x) dx = F(u(x)) + C .$$

Integration by parts:

$$\int u'(x)v(x) dx = u(x)v(x) - \int u(x)v'(x) dx .$$

LAPTE: When we integrate by parts, we need to proceed in such a way that the resulting new integral is easier than the old one. If we classify the functions involved according to LAPTE (Logarithms; Algebraic/Polynomials; Trigonometric; Exponential), then we need to take the derivative of whichever factor is further to the left in LAPTE, and integrate whichever factor is further to the right. For example it's useful to take the derivative of the natural logarithm, because then we obtain $1/x$, which we can combine with polynomials or other powers of x if we have any. Trigonometric and exponential functions, on the other hand, remain equally complicated if we have to integrate them.

Table in (13.7)

Differentiation always works:**when you think you've found an integral,****check by differentiating!**