11. THE FUNDAMENTAL THEOREM OF CALCULUS

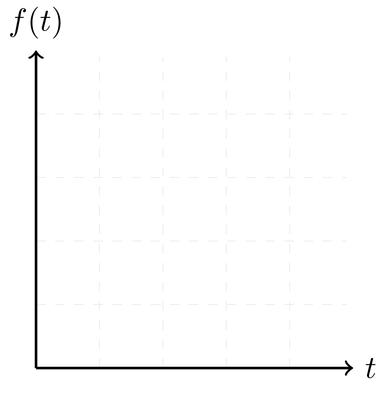
$\boxed{\text{(11.2) The definite integral as a function of its upper limit}}$

Here and in the following, I will denote an arbitrary (open, closed, etc.) interval. We consider a continuous function

$$f : I \to \mathbb{R}$$

and choose a fixed $a \in I$. For every $x \in I$ the definite integral

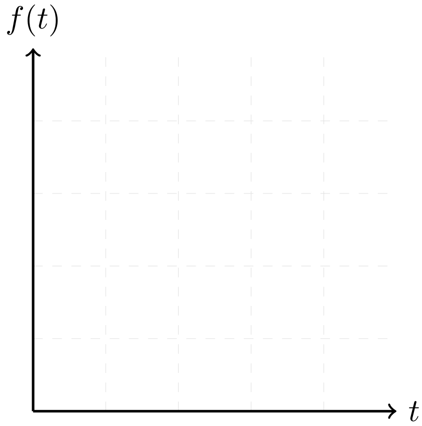
$\int_a^x f(t)\,dt$



is then defined in the sense of (10.2). In this way we obtain a new function

$$\boxed{\Phi : I \to \mathbb{R}\ , \quad \Phi(x) := \int_a^x f(t)\,dt\ .}$$

If $f(x) \geq 0$, then $\Phi(x)$ has a simple geometric interpretation as the area of the shaded region:



This is the case at any rate for $x > a$; according to (10.8) $\Phi(a) = 0$ ("vanishing" area), and for $x < a$, $\Phi(x)$ is the negative area.

Example: In (10.7) we have seen that the following holds:

$$\int_0^b x^2\,dx = \frac{b^3}{3}\,.$$

After a change of notation, in this special case we obtain a formula for the function Φ, namely

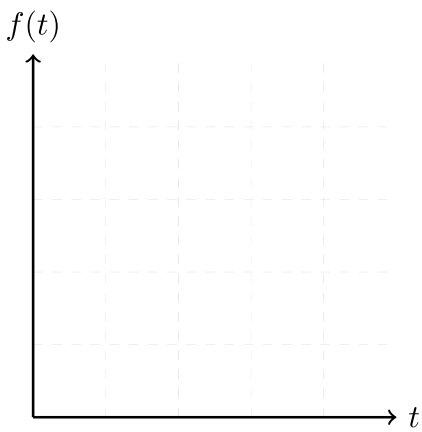
$$\Phi(x) = \int_0^x t^2\,dt = \frac{x^3}{3}\,.$$

Reverting to generality: If the function $\Phi(x) = \int_a^x f(t)\,dt$ is defined in this way, then the following extremely important statement is true:

Fact (I): Φ is differentiable on I and $\Phi'(x) = f(x)$ for all $x \in I$.

The only example of an integral for which we as yet know a mathematical expression is the one mentioned above. Here we have $f(x) = x^2$, $\Phi(x) = x^3/3$, and indeed we see that $\Phi'(x) = f(x)$. This corroborates Fact (I) above.

We will now give an intuitive justification for Fact (I):



(11.3) Antiderivatives

Building on Fact (I) from (11.2), we introduce a new and important concept:

Let $f : I \to \mathbb{R}$ be a function. By an *antiderivative* (or indefinite integral; German: Stammfunktion) of f we mean a function F such that:

$$F'(x) = f(x) \quad \text{for all} \quad x \in I\ .$$

Fact (I) can now also be formulated as follows: if f is continuous, then the function Φ given by

$$\Phi(x) = \int_a^x f(t)dt$$

is an antiderivative of f.

It follows that every continuous function f has (at least) one antiderivative, namely Φ. However, since the direct computation of a definite integral is in general so troublesome, this is at the moment only of theoretical interest.

The pivotal insight here is that one can also consider the problem from the opposite direction. In many cases an antiderivative can be given quite directly, simply by reversing the rules for the derivative. So for example $F(x) = \sin x$ is an antiderivative of $f(x) = \cos x$ (since $F'(x) = (\sin x)' = \cos x = f(x)$), $G(x) = x^2$ is an antiderivative of $g(x) = 2x$ (since $G'(x) = (x^2)' = 2x = g(x)$). Let's practice this approach right now:

We can make use of this state of affairs in order to compute definite integrals using their (somehow) directly identified antiderivatives. We will continue this train of reasoning using the first example. From the general theory we know that

$$\Phi(x) = \int_a^x \cos t \, dt$$

is an antiderivative of $f(x) = \cos x$. On the other hand, we have just seen directly that

$$F(x) = \sin x$$

is also an antiderivative of $\cos x$. If we were able to conclude now that these two antiderivatives Φ and F must be equal, i.e. that

$$\int_a^x \cos t \, dt = \sin x \quad \text{or in particular} \quad \int_a^b \cos t \, dt = \sin b$$

then we would have computed an integral — until now completely inaccessible — in the simplest way possible. Unfortunately this argument is not completely correct (but take heart, all is not lost!)

The reason that the argument conducted above is not correct is that a continuous function always has infinitely many antiderivatives. In fact:

Fact (II): If $F(x)$ is an antiderivative of $f(x)$, then for every real number C, $F(x)+C$ is also an antiderivative of $f(x)$.

This is a simple consequence of the fact that the derivative of any constant function is 0. Therefore indeed $(F(x) + C)' = F'(x) = f(x)$.

The converse of this statement is also true:

Fact (III): Let $F_1(x)$ and $F_2(x)$ be antiderivatives of $f(x)$. Then $F_1(x) - F_2(x) = C$ (for all x) for a suitable constant C, or written differently: $F_1(x) = F_2(x) + C$.

Proof:

Now we return to our previous example and consider it again with new courage. True, we cannot conclude that the two antiderivatives $\Phi(x)$ and $\sin x$ of $f(x) = \cos x$ are equal. However, we do know that they differ from each other only by a constant. There exists then a number C with

$$\Phi(x) = \int_a^x \cos t\, dt = \sin x + C\ .$$

How is this C to be determined? If we set $x = a$, then

$$\Phi(a) = \int_a^a \cos t\, dt = 0 = \sin a + C$$

and we find

$$C = -\sin a\ .$$

For all x, then, it holds that

$$\int_a^x \cos t\, dt = \sin x - \sin a$$

and particularly (with $x = b$)

$$\int_a^b \cos t\, dt = \sin b - \sin a\ .$$

Even more particularly, for example, we obtain the result

$$\int_0^{\pi/2} \cos t\, dt = \sin \frac{\pi}{2} - \sin 0 = 1\ .$$

This really is quite a triumph! The definition of this definite integral (with Riemann sums) was so complicated, and yet we are able to compute it in a very simple way.

If we interpret the last formula geometrically, we see that the area of the shaded region under the cosine curve is $= 1$!

Let's try the same approach again: we want to calculate

$$\int_a^b t^3\, dt\ .$$

(11.4) The fundamental theorem of calculus

The following result is extremely important for the computation of integrals.

<u>The Fundamental Theorem of Calculus</u>:

Let the function f be defined and continuous on the interval I, and let F be an arbitrary antiderivative of f. Then for all $a, b \in I$:

$$\int_a^b f(x)\,dx = F(b) - F(a)\ .$$

The proof of this theorem follows the reasoning already presented. According to "Fact (I)" of (11.2), $\Phi(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$. According to "Fact (III)" of (11.3), then, Φ and F only differ by a constant C:

$$\int_a^x f(t)\,dt = F(x) + C \quad \text{for all} \quad x \in I\ .$$

Since $\Phi(a) = 0$, we have $F(a) + C = 0$, i.e. $C = -F(a)$. For $x = b$ we obtain

$$\int_a^b f(t)\,dt = F(b) - F(a)\ .$$

Aside from the choice of integration variable (which does not signify) this is exactly the claim of the theorem. It should be emphasises that the formulation of the Fundamental Theorem is valid for $a > b$ as well as for $a < b$ (and trivially for $a = b$).

Observe that the Fundamental Theorem establishes a close relationship between the derivative and the integral, one which was not obvious a priori (hence the two coinciding names "*anti*derivative" and "indefinite integral").

For the frequently occurring difference $F(b) - F(a)$ we will often use the shorthand

$$\boxed{F(b) - F(a) = F(x)\Big|_a^b \ \text{or also} \ \Big[F(x)\Big]_a^b\ .}$$

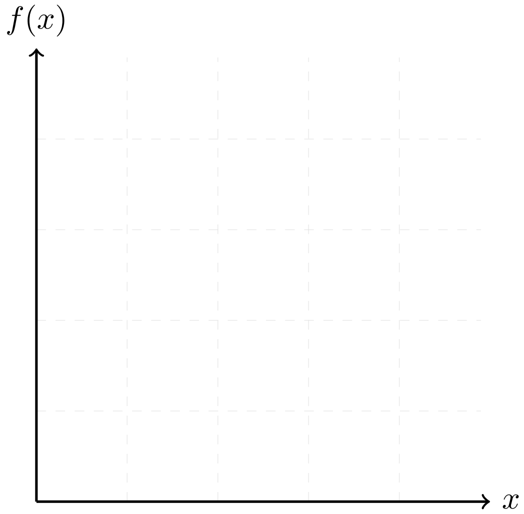
For example:

$$x^2\Big|_a^b = b^2 - a^2\ .$$

In the next chapter we will apply the Fundamental Theorem systematically. We conclude here with one more very simple example, which also serves to explain some notation. We consider the constant function $f(x) = 1$. Instead of $\int_a^b 1\, dx$ we may simply write $\int_a^b dx$. Clearly $F(x) = x$ is an antiderivative of $f(x)$, and so we obtain

$$\int_a^b dx = x\Big|_a^b = b - a\ .$$

Geometrically this is the area of the rectangle with height 1 on the interval bounded by a and b (if $a < b$; otherwise $b - a < 0$, and we get the negative area).



Important:

1. Next, read the corresponding chapter of the book yourself.

2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.

3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).

4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)

5. Solve the corresponding problems from past exams in the course archive.

Lessons learnt:

In this form, an antiderivative always exists: $\Phi(x) := \int_a^x f(t)$

Fact (I): Φ is differentiable on I and $\Phi'(x) = f(x)$ for all $x \in I$.

Let $f : I \to \mathbb{R}$ be a function. By an *antiderivative* (or indefinite integral; German: Stammfunktion) of f we mean a function F such that:

$$F'(x) = f(x) \quad \text{for all} \quad x \in I \ .$$

Fact (I) can now also be formulated as follows: if f is continuous, then the function Φ given by

$$\Phi(x) = \int_a^x f(t)dt$$

is an antiderivative of f.

Therefore every continuous function f has (at least) one antiderivative, namely Φ.

Fact (II): If $F(x)$ is an antiderivative of $f(x)$, then for every real number C, $F(x)+C$ is also an antiderivative of $f(x)$.

The converse of this statement is also true:

Fact (III): Let $F_1(x)$ and $F_2(x)$ be antiderivatives of $f(x)$. Then $F_1(x) - F_2(x) = C$ (for all x) for a suitable constant C, or written differently: $F_1(x) = F_2(x) + C$.

<u>The Fundamental Theorem of Calculus</u>:

Let the function f be defined and continuous on the interval I, and let F be an arbitrary antiderivative of f. Then for all $a, b \in I$:

$$\int_a^b f(x)\,dx = F(b) - F(a) \ .$$

For the frequently occurring difference $F(b) - F(a)$ we will often use the shorthand

$$F(b) - F(a) = F(x)\Big|_a^b \quad \text{or also} \quad \Big[F(x)\Big]_a^b \ .$$