

This script is an extract, with gaps, from the book “Introduction to the Mathematical Treatment of the Natural Sciences I - Analysis” by Christoph Luchsinger and Hans Heiner Storrer, Birkhäuser Scripts. As a student you should buy the book as well and work your way through it completely during the course MAT 182. You are allowed to save this PDF and modify it as you like, for your own use during MAT 182. For further use outside of MAT 182, please contact the lecturer, Christoph Luchsinger of the University of Zürich, in advance. The copyright remains with Birkhäuser!

## 10. THE DEFINITE INTEGRAL

### (10.2) Definition of the definite integral

As alluded to in (9.6), our plan here is to introduce a construction which generalises the process we investigated in the examples of Chapter 9.

To this end, consider a closed interval  $[a, b]$  and a *continuous function*

$$f : [a, b] \rightarrow \mathbb{R}, \quad (a < b).$$

We proceed in a series of steps:

#### Step 1

We divide the interval  $[a, b]$  into  $n$  *subintervals* by choosing a set of endpoints

$$a = x_0 < x_1 < x_2 < \dots < x_{i-1} < x_i < \dots < x_n = b.$$

(We may also speak of a *subdivision* of the interval  $[a, b]$ .) The  $i$ -th subinterval  $[x_{i-1}, x_i]$  has length  $\Delta x_i = x_i - x_{i-1}$ . (It is not necessary that all the subintervals have the same length.)

#### Step 2

In each subinterval  $[x_{i-1}, x_i]$  we choose an *intermediate point*  $\xi_i$ :

$$x_{i-1} \leq \xi_i \leq x_i.$$

#### Step 3

With the quantities thus chosen, we form the sum

$$\sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^n f(\xi_i)\Delta x_i.$$

Such a sum is called a *Riemann sum* (after B. RIEMANN, 1826–1866).

Step 4

Now we make the subintervals smaller and smaller (i.e. we let  $\Delta x_i \rightarrow 0$ ). As this happens, the number  $n$  of subintervals automatically increases ( $n \rightarrow \infty$ ).

Through this process the Riemann sum approaches a limit. This limit is called the *definite integral of  $f$  from  $a$  to  $b$*  and is denoted by

$$\boxed{\int_a^b f(x) dx .}$$

Formally written down:

$$\boxed{\int_a^b f(x) dx = \lim_{\Delta x_i \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i .}$$

In this way the definite integral is introduced as a general mathematical construction. A comparison with (9.2) through (9.5) will show that the observations we made there are concrete applications of the abstract concept of the integral (see also (10.5)).

\* Important observations in the textbook (read at home): (10.3), (10.5), and (10.6a)

\* for instance, answers to “what exactly happens ‘at infinity’?”: no esoterica required, just study mathematics to learn the whole story.

\*  $\int_a^b f(x) dx$  is a number, if  $a$  and  $b$  are given. An integral is called a *definite integral* if the limits of integration are given.

\* The variable of integration can be arbitrarily changed and disappears in the calculation; the following expressions are all equal, no matter whether we work with  $x, u$ , or  $s$ :

$$\int_a^b f(x) dx = \int_a^b f(u) du = \int_a^b f(s) ds.$$

\* Please NEVER write

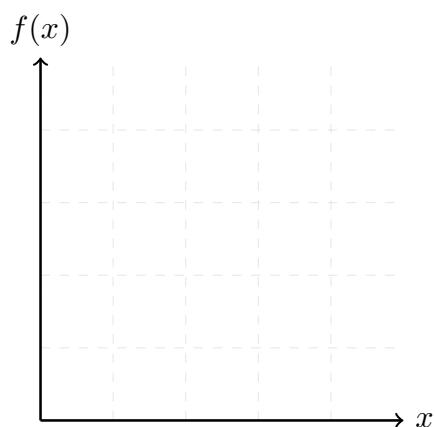
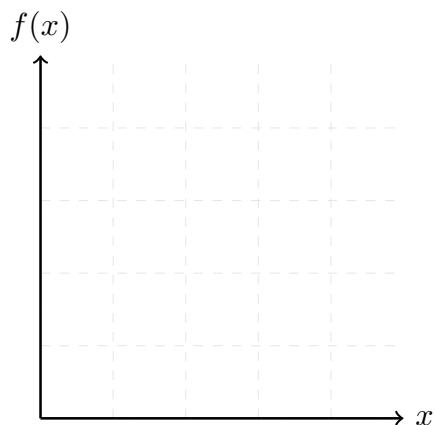
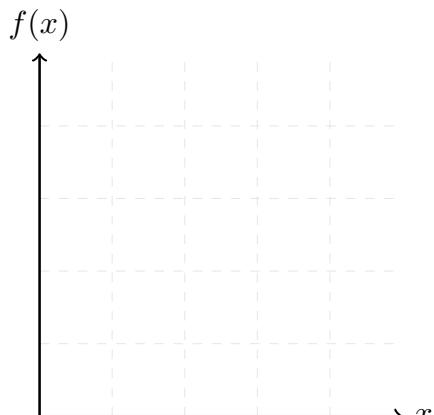
$$\int_a^x f(x) dx$$

The *limit* of integration should NEVER be the *variable* of integration!!!!

## (10.6) Further examples of applications of integrals

This section complements the examples in Chapter 9. Once again we aim to show that the concept of the integral is compellingly necessary to the description of certain phenomena.

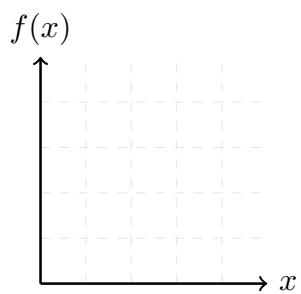
b) Arc length (length along a curve)



Arc length, continued:

(10.7) Computation of definite integrals: an example

Until now we have not explicitly computed a single integral; we will do that *systematically* later. Right now let's just compute the integral of  $x^2$  "by hand". A related article of mine (in German) about a famous formula by Gauss: <https://schweizermonat.ch/der-kleine-gauss-laesst-einem-keine-ruhe>.



Integral of  $x^2$  “by hand”, continued:

(10.8) Some rules for the definite integral

a) Sums, differences, multiples

The following rules hold for the definite integral:

$$(1) \int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$(2) \int_a^b cf(x) dx = c \int_a^b f(x) dx .$$

b) Exchanging the limits of integration

The integral  $\int_a^b f(x) dx$  was defined under the explicit condition that  $a < b$ . For  $a \geq b$  we may *define*, in broad agreement with intuitive understanding:

$$(3) \int_a^a f(x) dx = 0$$

$$(4) \int_a^b f(x) dx = - \int_b^a f(x) dx \quad \text{for } b < a .$$

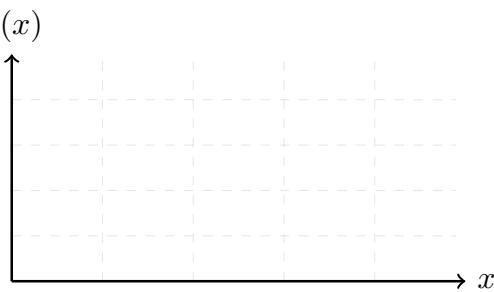
(Exchanging the limits of integration results in a sign change!)

c) Subdividing the interval

Let  $c \in [a, b]$ . Then

$$(5) \quad \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx .$$

If one considers the special case in which the integral represents an area, then this result is geometrically clear (it can however also be proven purely computationally):



Additionally, the above formula is also valid for  $a, b, c$  in arbitrary order, i.e. for  $c < b < a$  etc. This can be proven by consideration of the various possible cases.

**Important:**

1. Next, read the entire corresponding chapter of the book yourself.
2. Solve at least 5 of the end-of-chapter exercises and compare your answers with those in the back of the book. If needed, solve more than 5.
3. Now you are ready to attend an exercise session. Print the relevant part of the exercise script *in advance*, read through the exercises there *in advance* and start to think about them yourself (for example, how you might approach solving them).
4. Next, solve the problems on the worksheet. Always try them first yourself. If it doesn't work, try with a tip from a fellow student. If it still doesn't work, look at another student's solution, wait 1 hour and try to solve it again from your own head. If none of that works: follow another student's solution (but be sure you understand it - in particular see that you aren't copying someone else's mistakes!)
5. Solve the corresponding problems from past exams in the course archive.

**Lessons learnt:**

$$\int_a^b f(x) dx = \int_a^b f(u) du = \int_a^b f(s) ds.$$

Please NEVER write

$$\int_a^x f(x) dx$$

The *limit* of integration should NEVER be the *variable* of integration!!!!

Arc length (length along a curve)  $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$

The following rules hold for the definite integral:

$$(1) \quad \int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$(2) \quad \int_a^b cf(x) dx = c \int_a^b f(x) dx .$$

For  $a \geq b$  we define:

$$(3) \quad \int_a^a f(x) dx = 0$$

$$(4) \quad \int_a^b f(x) dx = - \int_b^a f(x) dx \quad \text{für } b < a .$$

(Exchanging the limits of integration results in a sign change!)

Let  $c \in [a, b]$ . Then

$$(5) \quad \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx .$$